Mp^c Structures and Applications

Paul Lee Robinson

Thesis submitted for the degree of

Doctor of Philosophy

July 1984

Mathematics Institute
University of Warwick
Coventry CV4 7AL
ENGLAND.

To my parents

Summary.

Mp^c is the group of all automorphisms of an irreducible Weyl system which project to the symplectic group, and contains the metaplectic group as commutator subgroup. Whereas to be metaplectic is a topological restriction, every symplectic vector bundle admits Mp^c structures.

We investigate the properties of Mp^c structures and their associated bundles of symplectic spinors. As applications of these concepts, we present a geometric quantization scheme extending those due to Kostant & Hess and indicate their implications for the symbol theory of operators.

Contents.

Acknowledgements.

I would like to express sincere gratitude to my supervisor John Rawnsley for his astute guidance throughout the course of my research.

I am deeply grateful to my family and friends for their support and interest.

My thanks go also to Peta McAllister for her careful typing of the manuscript.

The financial assistance of an S.E.R.C. studentship (1980-1983) is duly acknowledged.

Declaration.

As indicated in the text, portions of John Rawnsley's account [Ry5] of the Bargmann-Segal model are incorporated into §§1,3,4,5,6 alongside work of my own.

The contents of §§8,9 are due in part to John Rawnsley, in part to myself; these sections should therefore be regarded as joint work.

Except where expository or otherwise attributed, the remainder of this thesis is my own work.

§0. Introduction.

Motivation.

The symplectic group $Sp = Sp(V,\Omega)$ of a symplectic vector space (V,Ω) admits a unique connected double cover called the metaplectic group, Mp, which carries an infinite-dimensional unitary representation known as the metaplectic (also harmonic or Segal-Shale-Weil) representation. Mp can be realized as a group of automorphisms of an irreducible Weyl system (thus: an irreducible unitary representation W of the Heisenberg group N on a Hilbert space $\mathbb{H}$, having non-trivial central character) with the metaplectic representation as the natural consequence. The full automorphism group of W projecting to Sp is a central circle extension of Sp which we denote by Mp^c (this notation being suggested by the analogous way in which $Spin^c$ arises as automorphisms of a Clifford system); Mp^c thus naturally carries the metaplectic representation. Mp is contained in Mp^c as a distinguished normal subgroup : the kernel of a generator η of the infinite cyclic group of unitary characters of Mp^c. This much is well known and quite standard; see Howe [He] and Weil [Wℓ].

The conjugate-linear dual $E' = \mathbb{H}_{-\infty}$ of the Fréchet space $E = \mathbb{H}_{\infty}$ of smooth vectors for the representation $W: N \to \mathrm{Aut}\,\mathbb{H}$ is naturally a module for the complexified Heisenberg algebra $n^{\mathbb{C}}$. Under the (complexified differentiated) representation $\dot{W}^{\mathbb{C}} : n^{\mathbb{C}} \to \mathrm{End}\, E'$, a positive polarization Γ of (V,Ω) annihilates precisely a complex line $(E')^{\Gamma} \subset E'$: the vacuum state for Γ . If Γ_1 and Γ_2 are transverse positive polarizations then there is a canonical nonsingular sesquilinear pairing of the vacuum states $(E')^{\Gamma_1}$ and $(E')^{\Gamma_2}$ into $\mathbb{C}$. See Kostant [Kt2] [Kt3] and Rawnsley [Ry5].

These symplectic spinors E' have found application in several directions. Kostant [Kt2] [Kt3] introduced them as a tool in geometric quantization, where the vacuum state pairing facilitates the construction of Hilbert spaces. Boutet de Monvel & Guillemin [BG] have applied them to the development of a symbol theory for Toeplitz operators.

The principle behind these applications to a symplectic manifold (X,ω) is as follows. The symplectic frame bundle (X,ω) is lifted to a metaplectic structure (thus: a double cover by a principal Mp bundle) to which is associated a bundle of symplectic spinors via the metaplectic representation on E' . That (X,ω) admit a metaplectic structure is a topological restriction. The symplectic form ω determines a characteristic (Chern) class $c_1(\omega) \in H^2(X;\mathbb{Z})$ whose mod_2 reduction is the second Stiefel-Whitney class $w_2(X)$. It turns out that (X,ω) admits metaplectic structures iff $c_1(\omega)$ is even (which compares with the orthogonal case, where an oriented Riemannian manifold M admits Spin structures iff $w_2(M) = 0$).

As observed by several authors (Forger & Hess [FH], Rawnsley and Plymen [Pn]) there is no obstruction to the existence of Mp^c structures for (X,ω) : lifts of the symplectic frame bundle to structure group Mp^c always exist. Since Mp^c carries the metaplectic representation it is therefore possible to define symplectic spinors on any symplectic manifold.

It is now natural to ask whether use can be made of the symplectic spinors $E'(P)$ associated to an Mp^c structure P as was made of those defined by a metaplectic structure. One of our aims is to answer this question in the affirmative. Indeed, we shall see that in several respects Mp^c structures appear to be rather more natural objects than metaplectic structures.

We consider in particular geometric quantization of (X,ω) ; as general references we cite Guillemin & Sternberg [GS] and Woodhouse [We].

The full Kostant scheme requires both that the de Rham class $[\omega]$ be integral and that the Chern class $c_1(\omega)$ be even; the integrality of $[\omega]$ guarantees the existence of a prequantum line bundle for (X,ω) (thus: a Hermitian line bundle on X with metric connexion of curvature $-2\pi i\omega$) to provide a prequantization module for the Poisson algebra of functions on X , and a metaplectic structure for (X,ω) provides a means of constructing Hilbert spaces via the vacuum state pairing. See Kostant [Kt3].

It was observed by Hess [Hs] that in order to quantize (X,ω) it is not necessary to assume the separate existence of prequantum line bundle and metaplectic structure. He showed that by means of an Mp^c structure equipped with a special $u(1)$-valued form it is possible to quantize (X,ω) under the less restrictive constraint that the real cohomology class $[\omega] + \frac{1}{2} c_1(\omega)^{\mathbb{R}}$ be integral. The Hess scheme is thus able to deal in a uniform manner with the complex projective spaces $\mathbb{C}P_N$ (as is the scheme due to Czyz [Cz]); the even spaces $\mathbb{C}P_{2n}$ are not amenable to treatment by the Kostant scheme, since they admit neither prequantum line bundles nor metaplectic structures. Hess quantizes (X,ω) directly, without passing through a prequantization stage, and relative to a pair of polarizations rather than a single polarization; moreover, he gives no means whereby the results of quantizing relative to different polarizations may be compared.

We here propose a geometric quantization scheme which brings the Hess approach in line with the Kostant formalism. We use the symplectic spinors defined by an Mp^c structure as prequantization module, and the natural vacuum state (or half-form) pairing enables us both to construct Hilbert spaces for quantization and to compare the quantizations arising from different polarizations.

Regarding the application of Mp^c structures to a symbol theory for operators we refer to our final chapter which is itself introductory in character.

Preview.

Let us briefly outline the contents of this thesis.

§1 is an (admittedly rather biased) account of symplectic geometry; most of this is quite standard though some appears to be new. For a less partisan approach we refer to Weinstein [Wn].

In §§2,3,4,5,6,7 we present the group Mp^c and its metaplectic representation in some detail. We introduce the context in §2, make technical preparations in §3, and then proceed to discuss the metaplectic representation in terms of the Bargmann-Segal (BS) model on Fock space. The BS model has appeared in several places and in varying depth; see Bargmann [Bn1] [Bn2], Cartier [Cr], Itzykson [In], Rawnsley [Ry5], Segal [Sℓ], Sternberg & Wolf [Sw], ... As noted by Rawnsley [Ry5] the BS model is especially well-suited to making explicit computations with the metaplectic representation - in particular when dealing with vacuum states and their pairing.

The general theory of Mp^c structures is taken up in §8. We establish the unconditional existence of Mp^c structures for a symplectic vector bundle $(E \to X,\omega)$ and introduce a twisting $(Y,P) \mapsto P^Y$ of Mp^c structures P by principal $U(1)$ bundles Y which enables us to investigate the structure of the space $T(E,\omega)$ of equivalence classes of Mp^c structures for (E,ω). It turns out that $T(E,\omega)$ is naturally an abelian group isomorphic to $H^2(X;\mathbb{Z})$: at the level of equivalence classes, a choice of Mp^c structure for (E,ω) naturally corresponds to a choice of complex line bundle on X. We demonstrate that Mp^c structures for (E,ω) always induce Mp^c structures for the symplectic normal bundle $(D^{\perp}/D,\omega_D)$ of an isotropic subbundle D of (E,ω). Concluding remarks compare Mp^c structures with metaplectic structures (favourably).

§9 concerns bundles of symplectic spinors on (E,ω) defined by an Mp^c structure P for (E,ω). A positive polarization F of (E,ω) determines a complex line bundle $\mathcal{E}'(P)^F \subset \mathcal{E}'(P)$ of vacuum states. If Y is the unitary frame bundle of a Hermitian line bundle L then $\mathcal{E}'(P^Y)^F$ is canonically isomorphic to $\mathcal{E}'(P)^F \otimes L$. The half-form bundle $\mathcal{E}'(P)^F \otimes K^F$ is a canonical square-root of $P(\eta) \otimes K^F$, where $P(\eta)$ is the Hermitian line bundle associated to P via η and K^F is the canonical bundle of F. If (F,G) is a transverse pair of positive polarizations of (E,ω) then $\mathcal{E}'(P)^F$ and $\mathcal{E}'(P)^G$ pair naturally into the product complex line bundle $\underline{\mathbb{C}} = X \times \mathbb{C}$. The passage of Mp^c structures to symplectic normals enables us to define a nonsingular sesquilinear pairing of the half-form bundles $\mathcal{E}'(P)^F \otimes K^F$ and $\mathcal{E}'(P)^G \otimes K^G$ into the bundle $\mathcal{D}^{-1}(D)$ of inverse densities on D whenever (F,G) is a (regular) pair of positive polarizations of (E,ω) with $F \cap \bar{G} = D^{\mathbb{C}}$ for an isotropic subbundle D of (E,ω).

§10 is essentially a reformulation of the Hess approach, and sets up the prequantization data for our proposed quantization scheme. The fundamental unit of data is a prequantized Mp^c structure for (X,ω) : an Mp^c structure P equipped with a special $u(1)$-valued (prequantum) form γ . γ corresponds precisely with a metric connexion ∇^{γ} (of curvature $- 4\pi i\omega$) in the Hermitian line bundle $P(\eta)$.

Prequantization we present in §11. The approach we adopt is to take the sections $\Gamma(X;E'(P))$ as representation space for the Poisson algebra $C(X)$, the prequantization itself $(\delta:C(X) \to \mathrm{End}\,\Gamma(X;E'(P)))$ arising from the prequantum form γ . Although our approach to prequantization seems at first sight to be unwieldy, it leads to a natural development of quantization as can be seen in §12. If F is a positive polarization of (X,ω) then prequantization δ restricts naturally to give operators on $E'(P)^F$ defined for functions on X whose Hamiltonian flows preserve F . Tensoring with Lie derivative in K^F then gives a representation δ^F of these functions on the sapce of polarized (or: covariant constant along F) sections of the half-form bundle $E'(P)^F \otimes K^F$. This quantization δ^F on $E'(P)^F \otimes K^F$ squares up on $P(\eta) \otimes K^F$ to give the Kostant prequantization of $(X,2\omega)$ on its prequantum line bundle $(P(\eta),\nabla^{\gamma})$ tensored with Lie differentiation in K^F ; this observation turns out to be rather useful in practice. The pairing of half-forms allows us to construct Hilbert spaces for quantization and compare the quantizations arising from different polarizations.

The test of our proposed quantization scheme comes in §§13,14, where we discuss specific examples. §13 concerns the special case of a linear symplectic manifold and gives a convenient local picture of the scheme. §14 concerns the physically more interesting case of a complex projective space - the orbit space of the energy surface of a harmonic oscillator.

Our final section, §15, considers possible developments of the subject matter of this thesis. In particular, a procedure is suggested whereby Mp^c structures can be employed in the construction of symbols for operators.

Reasons of economy have necessitated the omission of a number of related topics of interest; we hope that these will appear elsewhere in due course.

Conventions.

We assume a knowledge of the elements of differential geometry - in particular we assume a familiarity with principal and associated bundles and connexions therein; see Kobayashi & Nomizu [KN].

Let G be a Lie group. By a character of G we shall mean a Lie group morphism $\chi : G \to \overset{\circ}{\mathbb{C}}$ into the multiplicative group $\overset{\circ}{\mathbb{C}}$ of nonzero complex numbers; we shall say that χ is unitary iff it takes values in the unit circle $U(1)$.

Let P be a (right) principal G bundle over the manifold X . If A is a (left) G space then we may denote by $A(P)$ the bundle associated to P with typical fibre A ; the total space of $A(P)$ is the quotient of $P\times A$ by the G-action $(p,a)\cdot g = (p.g, g^{-1}.a)$ for $p \in P$, $a \in A$, $g \in G$, and we have a natural projection

$$P\times A \to A(P) : (p,a) \mapsto [p,a] \quad .$$

We may view $p \in P_x$ as an isomorphism

$$p : A \to A(P)_x : a \mapsto [p,a]$$

with the property

$$(p.g)(a) = p(g.a)$$

for $a \in A$ and $g \in G$.

If χ is a (unitary) character of G then $P(\chi)$ will denote the (Hermitian) complex line bundle associated to P with typical fibre $\mathbb{C}$ via the action of G on $\mathbb{C}$ determined by χ .

§1. Symplectic Geometry.

In this section we give a brief review of those aspects of symplectic geometry which we regard as fundamental to our work. We do this under five subsections; the first four subsections are essentially algebraic in content and the last deals with bundles and manifolds. Our treatment is by no means intended to be exhaustive; indeed it is specially tailored to our needs. Further topics will be introduced as they are required.

Symplectic Vector Spaces : Sp and sp .

Let (V,Ω) be a real symplectic vector space of dimension $2m$ - thus, V is a $2m$-dimensional real vector space and Ω is a symplectic (by definition, nonsingular alternating real bilinear) form on V. As is customary, we shall frequently omit explicit reference to Ω unless confusion is likely to arise.

The symplectic group $Sp(V,\Omega)$ is the group of all real linear automorphisms g of V which preserve Ω in the sense that

$$\Omega(gv_1,gv_2) = \Omega(v_1,v_2) \tag{1.1}$$

for $v_1,v_2 \in V$. $Sp(V,\Omega)$ is a connected semisimple Lie group with centre $\{\pm I\}$ where I denotes the identity map on V. The Lie algebra of $Sp(V,\Omega)$ is naturally embedded in End V as the symplectic algebra $sp(V,\Omega)$ which consists of all real linear endomorphisms ξ of V satisfying

$$\Omega(\xi v_1, v_2) + \Omega(v_1, \xi v_2) = 0 \tag{1.2}$$

for $v_1, v_2 \in V$.

The (Ω-) orthocomplement of the subspace L of V is the subspace L of V defined by

$$L^{\perp} = \{v \in V \mid \ell \in L \Rightarrow \Omega(v,\ell) = 0\} . \tag{1.3}$$

We say that L is symplectic iff $L \cap L^{\perp} = 0$, isotropic iff $L \subset L^{\perp}$, coisotropic iff $L \supset L^{\perp}$, and Lagrangian iff $L = L^{\perp}$. The Lagrangian subspaces of (V,Ω) are precisely the maximally isotropic subspaces; dually, they coincide with those subspaces which are minimally coisotropic.

As a subgroup of the group $G\ell(V)$ of all real linear automorphisms of V , $Sp(V,\Omega)$ has a natural action on the space of all subspaces of V , given by

$$g.L = \{g\ell \mid \ell \in L\} \tag{1.4}$$

for $g \in Sp(V,\Omega)$ and L a subspace of V . We note that the isotropic subspaces of V form a union of orbits for this action, with dimension parametrizing the individual orbits. Let $Sp(V,\Omega;L)$ denote the stabilizer of the subspace L of V relative to this action, and observe that

$$Sp(V,\Omega;L^{\perp}) = Sp(V,\Omega;L) . \tag{1.5}$$

A Hilbert structure for (V,Ω) is a real linear automorphism J of V such that

$$J^2 = - I \tag{1.6}$$

and such that the bilinear form

$$V \times V \to \mathbb{R} : (v_1,v_2) \mapsto \Omega(Jv_1,v_2) \tag{1.7}$$

is symmetric and positive-definite.

$$\langle v_1,v_2\rangle = \Omega(Jv_1,v_2) + i\Omega(v_1,v_2) \tag{1.8}$$

then defines a Hermitian inner product on the complex vector space V^J (which has V as underlying real vector space and on which multiplication by i is given by the action of J). Thus, a choice of Hilbert structure J for (V,Ω) naturally gives rise to a complex Hilbert space $(V^J,\langle\cdot,\cdot\rangle)$.

The unitary group

$$U(V,\Omega;J) = \{g \in Sp(V,\Omega) \mid gJ = Jg\} \tag{1.9}$$

of $(V^J,\langle\cdot,\cdot\rangle)$ is a maximal compact subgroup of $Sp(V,\Omega)$; moreover, all maximal compact subgroups of $Sp(V,\Omega)$ arise in this way from Hilbert structures for (V,Ω) .

As set down in detail by Rawnsley [Ry5], a choice of Hilbert structure for (V,Ω) determines a parametrization of $Sp(V,\Omega)$ to which we now turn our attention.

Fix a Hilbert structure J for (V,Ω) . We can express each real linear endomorphism g of V as a sum

$$g = C_g + A_g \tag{1.10}$$

where

$$C_g = \tfrac{1}{2}(g - JgJ) \tag{1.11}$$

commutes with J (or, is J-linear) and

$$A_g = \tfrac{1}{2}(g + JgJ) \tag{1.12}$$

anticommutes with J (or, is J-antilinear).

It turns out that if $g \in Sp(V,\Omega)$ then the complex linear endomorphism C_g of V^J is invertible; thus we may define

$$Z_g = C_g^{-1} A_g \tag{1.13}$$

so that

$$g = C_g(I + Z_g) \ . \tag{1.14}$$

Some of the basic properties of the decomposition (1.14) are as follows.

<u>Proposition 1.1</u> : If $g \in Sp(V,\Omega)$ then

(i) $C_{g^{-1}} = C_g^*$ (*:<·,·>-adjoint) (1.15)

(ii) $Z_{g^{-1}} = -C_g Z_g C_g^{-1}$ (1.16)

(iii) $v_1, v_2 \in V \Rightarrow \langle v_1, Z_g v_2\rangle = \langle v_2, Z_g v_1\rangle$ (1.17)

(iv) $C_g(I-Z_g^2)C_g^* = I$ (1.18)

(v) $v \in V\setminus\{0\} \Rightarrow \langle (I-Z_g^2)v, v\rangle > 0$ (1.19)

<u>Proof</u>: See Rawnsley [Ry5]. □

Let $\bar{\mathcal{D}} = \bar{\mathcal{D}}(V,\Omega;J)$ denote the set of all real linear endomorphisms Z of V which satisfy

(i) $ZJ + JZ = 0$ (1.20)

(ii) $v_1, v_2 \in V \Rightarrow \langle v_1, Zv_2 \rangle = \langle v_2, Zv_1 \rangle$ (1.21)

(iii) $v \in V \Rightarrow \langle (I-Z^2)v, v \rangle \geq 0$ (1.22)

and denote by $\mathcal{D} = \mathcal{D}(V,\Omega;J)$ the subset of $\bar{\mathcal{D}}$ consisting of those Z for which strict inequality holds in (iii) whenever $v \neq 0$. $\bar{\mathcal{D}}$ (respectively, $\mathcal{D}$) is our version of the closed (respectively, open) Siegel domain for $Sp(V,\Omega)$.

<u>Proposition 1.2</u> : If $g \in Sp(V,\Omega)$ then the pair (C_g, Z_g) lies in $G\ell(V^J) \times \mathcal{D}(V,\Omega;J)$ and we have $C_g(I-Z_g^2)C_g^* = I$. Conversely, if the pair (C,Z) in $G\ell(V^J) \times \mathcal{D}(V,\Omega;J)$ satisfies

$$C(I-Z^2)C^* = I \qquad (1.23)$$

then $g = C(I+Z)$ lies in $Sp(V,\Omega)$.

<u>Proof</u>: See Rawnsley [Ry5].

□

The preceding result gives the promised parametrization of $Sp(V,\Omega)$; the next describes the group multiplication of $Sp(V,\Omega)$ in terms of this parametrization.

<u>Proposition 1.3</u> : If $g_1, g_2 \in Sp(V,\Omega)$ then

$$\text{(i)} \quad C_{g_1 g_2} = C_{g_1}(I - Z_{g_1} Z_{g_2^{-1}}) C_{g_2} \tag{1.24}$$

$$\text{(ii)} \quad Z_{g_1 g_2} = C_{g_2}^{-1}(I - Z_{g_1} Z_{g_2^{-1}})^{-1}(Z_{g_1} - Z_{g_2^{-1}}) C_{g_2} . \tag{1.25}$$

<u>Proof</u>: See Rawnsley [Ry5].

□

Considering the symplectic algebra $sp(V,\Omega)$ we have

<u>Proposition 1.4</u> : The real linear endomorphism ξ of V lies in $sp(V,\Omega)$ iff

$$\langle C_\xi v_1, v_2 \rangle + \langle v_1, C_\xi v_2 \rangle = 0 \tag{1.26}$$

$$\langle v_1, A_\xi v_2 \rangle = \langle v_2, A_\xi v_1 \rangle \tag{1.27}$$

whenever $v_1, v_2 \in V$.

<u>Proof</u>: Routine algebra; we omit the details, merely remarking that the criterion (1.2) for ξ to lie in $sp(V,\Omega)$ may be rewritten

$$v_1, v_2 \in V \Rightarrow \Omega(v_1, \xi v_2) = \Omega(v_2, \xi v_1) .$$

□

<u>Remark 1.5</u> : Observe that for $sp(V,\Omega)$ there is no compatibility requirement such as there was (see (1.23)) for $Sp(V,\Omega)$.

//

It is reasonable to ask for the way in which the parametrizations of $Sp(V,\Omega)$ and $sp(V,\Omega)$ are related by the exponential map

$$\exp : sp(V,\Omega) \to Sp(V,\Omega) .$$

The answer is the following.

<u>Proposition 1.6</u> : If $\xi \in sp(V,\Omega)$ then

$$\text{(i)} \quad \frac{d}{dt}\{C_{\exp t\xi}\}\Big|_{t=0} = C_\xi \tag{1.28}$$

$$\text{(ii)} \quad \frac{d}{dt}\{Z_{\exp t\xi}\}\Big|_{t=0} = A_\xi \ . \tag{1.29}$$

<u>Proof</u>: Evaluate on an arbitrary element of V and differentiate along one-parameter subgroups as indicated.

□

<u>Remark 1.7</u> : In the light of Proposition 1.6, we see that the criteria in Proposition 1.4 for membership of $sp(V,\Omega)$ are precisely the differentiated forms of the criteria in Proposition 1.2 for membership of $Sp(V,\Omega)$. //

<u>Symplectic Vector Spaces</u> : <u>Polarizations</u>.

Let $(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ denote the complex symplectic vector space obtained by the complexification of V and the $\mathbb{C}$-bilinear extension of Ω .

We denote conjugation in $V^{\mathbb{C}}$ relative to V by an upper bar : $\bar{\ }$.
A pseudo-Hermitian form H is defined on $V^{\mathbb{C}}$ by

$$H(v_1, v_2) = i\Omega^{\mathbb{C}}(v_1, \bar{v}_2) \tag{1.30}$$

for $v_1, v_2 \in V^{\mathbb{C}}$. An isomorphism

$$\beta : V^{\mathbb{C}} \to (V^{\mathbb{C}})^* \tag{1.31}$$

is defined in terms of interior multiplication $\lrcorner$ by

$$\beta(v) = v \lrcorner \Omega^{\mathbb{C}} \tag{1.32}$$

for $v \in V^{\mathbb{C}}$; thus

$$v_1, v_2 \in V^{\mathbb{C}} \Rightarrow \beta(v_1)(v_2) = \Omega^{\mathbb{C}}(v_1, v_2) \tag{1.33}$$

A polarization of (V, Ω) is a complex m-dimensional subspace Γ of $V^{\mathbb{C}}$ which satisfies

$$v_1, v_2 \in \Gamma \Rightarrow \Omega^{\mathbb{C}}(v_1, v_2) = 0 \quad . \tag{1.34}$$

Equivalently, Γ is a complex subspace of $V^{\mathbb{C}}$ whose annihilator

$$\Gamma^0 = \{\phi \in (V^{\mathbb{C}})^* \mid v \in \Gamma \Rightarrow \phi(v) = 0\} \tag{1.35}$$

equals $\beta(\Gamma)$. Thus, Γ is a (complex) Lagrangian subspace of

$(V^{\mathbb{C}},\Omega^{\mathbb{C}})$. It is natural to denote by $\mathrm{Lag}(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ the space of all polarizations of (V,Ω) .

Let $\Gamma \in \mathrm{Lag}(V^{\mathbb{C}},\Omega^{\mathbb{C}})$. The real part $\Gamma \cap \bar{\Gamma}$ of Γ is the complexification of an isotropic subspace of (V,Ω) . We denote by $r(\Gamma)$ the (complex) dimension of $\Gamma \cap \bar{\Gamma}$ and define the index $i(\Gamma)$ of Γ to be the (complex) dimension of a maximal subspace of Γ on which H is negative-definite. We say that Γ is of type $(r(\Gamma),i(\Gamma))$ and that Γ is positive iff $i(\Gamma) = 0$. The space of all positive polarizations of (V,Ω) will be denoted by $\mathrm{Lag}_+(V,\Omega)$.

The pair (Γ_1,Γ_2) of positive polarizations of (V,Ω) is said to be transverse iff

$$\Gamma_1 \cap \bar{\Gamma}_2 = 0 . \tag{1.36}$$

Positive polarizations have particularly pleasing intersection properties as demonstrated by

<u>Proposition 1.8</u> : If Γ_1 and Γ_2 are positive polarizations of (V,Ω) then

(i) $$\Gamma_1 \cap \bar{\Gamma}_2 = (\Gamma_1 \cap \bar{\Gamma}_1) \cap (\Gamma_2 \cap \bar{\Gamma}_2) = \bar{\Gamma}_1 \cap \Gamma_2 \tag{1.37}$$

(ii) $$r(\Gamma_2) = 0 \Rightarrow \Gamma_1 \cap \bar{\Gamma}_2 = 0 . \tag{1.38}$$

<u>Proof</u>: See Blattner [Br].

□

Remark 1.9 : (i) implies that $\Gamma_1 \cap \bar{\Gamma}_2$ is the complexification of an isotropic subspace of (V,Ω) . (ii) asserts that each type $(0,0)$ polarization is transverse to every positive polarization. //

$Sp(V,\Omega)$ acts naturally on the space $Lag(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ by complexification - thus

$$g \cdot \Gamma = \{g^{\mathbb{C}} \nu \mid \nu \in \Gamma\} \tag{1.39}$$

for $g \in Sp(V,\Omega)$ and $\Gamma \in Lag(V^{\mathbb{C}},\Omega^{\mathbb{C}})$. The orbits for this action are parametrized by type (r,i) ; in particular, this action restricts to give an action of $Sp(V,\Omega)$ on $Lag_+(V,\Omega)$ having $m+1$ orbits parametrized by dimension of real part.

We denote by $Sp(V,\Omega;\Gamma)$ the stabilizer of $\Gamma \in Lag(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ - thus

$$Sp(V,\Omega;\Gamma) = \{g \in Sp(V,\Omega) \mid g \cdot \Gamma = \Gamma\} \ . \tag{1.40}$$

Taking the (complex) determinant of the restriction to Γ yields a character

$$Det_\Gamma : Sp(V,\Omega;\Gamma) \to \overset{\circ}{\mathbb{C}} \tag{1.41}$$

of $Sp(V,\Omega;\Gamma)$; here, $\overset{\circ}{\mathbb{C}}$ denotes the multiplicative group of nonzero complex numbers.

We denote by $sp(V,\Omega)_\Gamma^{\mathbb{C}}$ the stabilizer of $\Gamma \in Lag(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ under the natural representation of $sp(V,\Omega)^{\mathbb{C}}$ on $V^{\mathbb{C}}$ - thus

$$sp(V,\Omega)_\Gamma^{\mathbb{C}} = \{\xi \in sp(V,\Omega)^{\mathbb{C}} | \xi\Gamma \subset \Gamma\} \tag{1.42}$$

$sp(V,\Omega)_\Gamma^{\mathbb{C}}$ naturally contains the complexification $sp(V,\Omega;\Gamma)^{\mathbb{C}}$ of the Lie algebra $sp(V,\Omega;\Gamma)$ of $Sp(V,\Omega;\Gamma)$. Taking the (complex) trace of the restriction to Γ yields a character

$$Tr_\Gamma : sp(V,\Omega)_\Gamma^{\mathbb{C}} \to \mathbb{C} \tag{1.43}$$

of $sp(V,\Omega)_\Gamma^{\mathbb{C}}$.

Fix a Hilbert structure J for (V,Ω). We shall see that J gives rise to a bijection (the Z-transform) from $Lag_+(V,\Omega)$ to $\mathcal{D}(V,\Omega;J)$.

Extend J to $V^{\mathbb{C}}$ by complex linearity and define complex linear endomorphisms $J\pm$ of $V^{\mathbb{C}}$ by

$$J_\pm = \tfrac{1}{2}(I \mp iJ) \tag{1.44}$$

$J\pm$ is projection of $V^{\mathbb{C}}$ on the $(\pm i)$-eigenspace of J. Define

$$\Gamma_0 := \text{im } J_+ \tag{1.45}$$

Γ_0 is a polarization of (V,Ω) of type $(0,0)$. Observe that

$$Sp(V,\Omega;\Gamma_0) = U(V,\Omega;J) . \tag{1.46}$$

If $\Gamma \in Lag_+(V,\Omega)$ then a $\mathbb{C}$-linear map

$$B_\Gamma : \Gamma \to V^J \tag{1.47}$$

is defined by requiring that

$$B_\Gamma(v_1+iv_2) = v_1 + Jv_2 \tag{1.48}$$

for $v_1,v_2 \in V$ with $v_1 + iv_2 \in \Gamma$.

<u>Proposition 1.10</u> : If $\Gamma \in Lag_+(V,\Omega)$ and $v \in V$ then there exists a unique $v_1 + iv_2 \in \Gamma$ such that $v = v_1 + Jv_2$.

<u>Proof</u>: B_Γ has kernel precisely

$$\Gamma \cap \{v + iJv \mid v \in V\} = \Gamma \cap \bar{\Gamma}_0 .$$

According to Proposition 1.8 this space is zero, whence B_Γ is injective. Since Γ and V^J have the same dimension, B_Γ is an isomorphism. The result follows.

□

If $\Gamma \in Lag_+(V,\Omega)$ then we define a real linear endomorphism Z_Γ of V by

$$Z_\Gamma(v) = v_1 - Jv_2 \tag{1.49}$$

for $v \in V$ and $v_1 + iv_2 \in \Gamma$ as in Proposition 1.10. We refer to Z_Γ as the Z-transform of Γ (relative to J) .

Proposition 1.11 : The taking of Z-transforms provides a bijection

$$Lag_+(V,\Omega) \to \mathcal{D}(V,\Omega;J) : \Gamma \mapsto Z_\Gamma \quad . \tag{1.50}$$

Proof: See Rawnsley [Ry5]. We remark only that the inverse transform of $Z \in \bar{\mathcal{D}}$ is $(I+Z)\Gamma_0 \in Lag_+(V,\Omega)$.

□

Relative to the natural action (1.39) of $Sp(V,\Omega)$ on $Lag_+(V,\Omega)$ we have

Proposition 1.12 : If $\Gamma \in Lag_+(V,\Omega)$ and $g \in Sp(V,\Omega)$ then

$$\text{(i)} \quad B_{g\cdot\Gamma} \circ g^{\mathbb{C}} = \{C_g(I+Z_gZ_\Gamma)\}\circ B_\Gamma \tag{1.51}$$

$$\text{(ii)} \quad Z_{g\cdot\Gamma} = C_g(Z_g+Z_\Gamma)(I+Z_gZ_\Gamma)^{-1}C_g^{-1} \quad . \tag{1.52}$$

Proof: A straightforward verification from the definitions. See Rawnsley [Ry5].

□

Remark 1.13 : From (1.51) we have

$$\mathrm{Det}_\Gamma g = \mathrm{Det}\{C_g(I+Z_g Z_\Gamma)\} \tag{1.53}$$

whenever $g \in Sp(V,\Omega;\Gamma)$.

//

For Lie algebra stabilizers we have the following analogue of (1.53).

Proposition 1.14 : If $\Gamma \in \mathrm{Lag}_+(V,\Omega)$ and

$\xi_1,\xi_2 \in sp(V,\Omega)$ with $\xi_1 + i\xi_2 \in sp(V,\Omega)_\Gamma^{\mathbb{C}}$

then

$$\mathrm{Tr}_\Gamma(\xi_1+i\xi_2) = \mathrm{Tr}\{(C_{\xi_1}+A_{\xi_1}Z_\Gamma) + J(C_{\xi_2}+A_{\xi_2}Z_\Gamma)\} \ . \tag{1.54}$$

Proof: By virtue of Proposition 1.10 we can express an arbitrary element of V as $v_1 + Jv_2$ for some unique $v_1 + iv_2 \in \Gamma$. Routine manipulations reveal that

$$(C_\xi + A_\xi Z_\Gamma)(v_1 + Jv_2) = \xi v_1 + J\xi v_2$$

whenever $\xi \in \mathrm{End}\ V$, whence B_Γ intertwines the $\mathbb{C}$-linear endomorphisms $(\xi_1 + i\xi_2)|_\Gamma$ (of Γ) and $\{(C_{\xi_1}+A_{\xi_1}Z_\Gamma) + J(C_{\xi_2}+A_{\xi_2}Z_\Gamma)\}$ (of V^J) . Taking traces, the result follows.

□

Symplectic Vector Spaces: Exterior Algebra.

In this subsection we discuss briefly some of the exterior algebra pertaining to symplectic vector spaces and polarizations.

The Liouville (volume) form

$$\lambda \in \Lambda^{2m}V^* \hookrightarrow \Lambda^{2m}(V^{\mathbb{C}})^* \tag{1.55}$$

is defined by

$$\lambda = (-1)^{\frac{m(m-1)}{2}} \frac{\Omega^m}{m!} \tag{1.56}$$

λ is clearly a basis vector for the complex line $\Lambda^{2m}(V^{\mathbb{C}})^*$.

Proposition 1.15 : A pseudo-Hermitian form

$$<\cdot,\cdot>_K : \Lambda^m(V^{\mathbb{C}})^* \times \Lambda^m(V^{\mathbb{C}})^* \to \mathbb{C} \tag{1.57}$$

is defined by

$$<k_1,k_2>_K\lambda = i^m(k_1 \wedge \bar{k}_2) \tag{1.58}$$

for $k_1,k_2 \in \Lambda^m(V^{\mathbb{C}})^*$.

Proof: Since λ is a basis vector for $\Lambda^{2m}(V^{\mathbb{C}})^*$ and the sesquilinear map

$$\Lambda^m(V^{\mathbb{C}})^* \times \Lambda^m(V^{\mathbb{C}})^* \to \Lambda^{2m}(V^{\mathbb{C}})^* : (k_1,k_2) \mapsto k_1 \wedge \bar{k}_2$$

is nonsingular, it is clear that (1.58) defines a nonsingular sesquilinear pairing $<\cdot,\cdot>_K$. A routine computation establishes

$$k_1,k_2 \in \Lambda^m(V^{\mathbb{C}})^* \Rightarrow <k_1,k_2>_K = \overline{<k_2,k_1>_K} \quad .$$

This completes the proof.

□

Remark 1.16 : Since $Sp(V,\Omega)$ is contained in the special linear group $S\ell(V)$ we have

$$<g\cdot k_1,g\cdot k_2>_K = <k_1,k_2>_K \tag{1.59}$$

for $g \in Sp(V,\Omega)$ and $k_1,k_2 \in \Lambda^m(V^{\mathbb{C}})^*$ (relative to the natural representation of $Sp(V,\Omega)$ on $\Lambda^m(V^{\mathbb{C}})^*$).

//

Let Γ be a polarization of (V,Ω) . The canonical line K^Γ of Γ is defined as the top exterior power of Γ^0 :

$$K^\Gamma = \Lambda^m\Gamma^0 \hookrightarrow \Lambda^m(V^{\mathbb{C}})^* \quad . \tag{1.60}$$

K^Γ is thus a one-dimensional complex subspace of $\Lambda^m(V^{\mathbb{C}})^*$

Proposition 1.17 : Let (Γ_1,Γ_2) be a pair of positive polarizations of (V,Ω) .

(i) If (Γ_1,Γ_2) is not transverse then $\langle\cdot,\cdot\rangle_K$ vanishes on $K^{\Gamma_1} \times K^{\Gamma_2}$.

(ii) If (Γ_1,Γ_2) is transverse then $\langle\cdot,\cdot\rangle_K$ is nonsingular on $K^{\Gamma_1} \times K^{\Gamma_2}$.

Proof: An exercise in elementary exterior algebra. The positivity restriction is not essential.

□

The choice J of Hilbert structure for (V,Ω) enables us to select canonical basis vectors in the canonical lines of positive polarizations. Let us now define these canonical vectors and briefly consider some of their properties.

It is routine to verify that the restriction of $\langle\cdot,\cdot\rangle_K$ to $K^{\Gamma_0} \times K^{\Gamma_0}$ is actually Hermitian (positive-definite), where $\Gamma_0 = \operatorname{im} J_+$ is as in (1.45). K^{Γ_0} thus becomes a one-dimensional complex Hilbert space. Fix a choice k_{Γ_0} of unit vector for K^{Γ_0} :

$$\langle k_{\Gamma_0}, k_{\Gamma_0}\rangle_K = 1 \quad . \tag{1.61}$$

Let $\Gamma \in \mathrm{Lag}_+(V,\Omega)$. Propositions 1.8 and 1.17(ii) allow us to assert the existence of a unique (basis) vector k_Γ of K^Γ such that

$$\langle k_\Gamma, k_{\Gamma_0}\rangle_K = 1 \tag{1.62}$$

k_Γ is our canonical basis vector for K^Γ .

Remark 1.18 : These vectors are not canonical in the strict sense; instead they are uniquely determined once we have selected an element of the principal $U(1)$ space of unit vectors for K^{Γ_0} .

//

Let $\Gamma \in \mathrm{Lag}_+(V,\Omega)$. Define an isomorphism (of complex lines)

$$\theta_\Gamma : K^\Gamma \to \Lambda^m V^J \tag{1.63}$$

by the prescription

$$\theta_\Gamma \circ \Lambda^m(\beta|_\Gamma) = \Lambda^m(B_\Gamma) \quad . \tag{1.64}$$

Proposition 1.19 : The element $\theta_\Gamma(k_\Gamma)$ of $\Lambda^m V^J$ is independent of the choice Γ of positive polarization of (V,Ω) .

Proof: Let us suppose that

$$\theta_{\Gamma_0}(k_{\Gamma_0}) = v_1 \wedge \ldots\ldots \wedge v_m$$

with $v_1,\ldots\ldots,v_m$ in V^J . Since the composite

$$V^J \xrightarrow{B_\Gamma^{-1}} \Gamma \hookrightarrow V^{\mathbb{C}} \xrightarrow{J_+} \Gamma_0 \xrightarrow{B_{\Gamma_0}} V^J$$

is the identity, it follows that

$$1 \le j \le m \implies B_\Gamma^{-1} v_j - B_{\Gamma_0}^{-1} v_j \in \overline{\Gamma_0}$$

whence

$$\theta_\Gamma^{-1}(v_1 \wedge \ldots\ldots \wedge v_m) \wedge \overline{k_{\Gamma_0}} = k_{\Gamma_0} \wedge \overline{k_{\Gamma_0}}$$

$$\langle \theta_\Gamma^{-1}(v_1 \wedge \ldots\ldots \wedge v_m), k_{\Gamma_0} \rangle_K = 1 \quad .$$

By uniqueness we conclude that

$$\theta_\Gamma^{-1}(v_1 \wedge \ldots\ldots \wedge v_m) = k_\Gamma \quad .$$

□

Relative to the natural representation of $Sp(V,\Omega)$ on $\Lambda^m(V^{\mathbb{C}})^*$, the canonical vectors transform as follows:

<u>Proposition 1.20</u> : If $\Gamma \in Lag_+(V,\Omega)$ and $g \in Sp(V,\Omega)$ then

$$g \cdot k_\Gamma = \mathrm{Det}\{C_g(I+Z_g Z_\Gamma)\} k_{g\cdot\Gamma} \quad . \tag{1.65}$$

<u>Proof</u>: A routine consequence of Propositions 1.12 (i) and 1.19.

□

The effect of the pairing $\langle\cdot,\cdot\rangle_K$ on our canonical vectors is as follows.

Proposition 1.21 : If (Γ_1,Γ_2) is a pair of positive polarizations of (V,Ω) then

$$\langle k_{\Gamma_1},k_{\Gamma_2}\rangle_K = \mathrm{Det}(I-Z_{\Gamma_2}Z_{\Gamma_1}) \quad . \tag{1.66}$$

Proof: $\langle k_{\Gamma_1},k_{\Gamma_2}\rangle_K$ and $\mathrm{Det}(I-Z_{\Gamma_2}Z_{\Gamma_1})$ obey the same transformation law under the action of $Sp(V,\Omega)$ on $Lag_+(V,\Omega)$ (see Proposition 1.20 and Rawnsley [Ry5]). Since

$$\begin{aligned}\Gamma \in Lag_+(V,\Omega) \Rightarrow \langle k_\Gamma,k_{\Gamma_0}\rangle_K &= 1\\ \& \quad \mathrm{Det}(I-Z_{\Gamma_0}Z_\Gamma) &= 1\end{aligned} \tag{1.67}$$

and the type $(0,0)$ polarizations of (V,Ω) form a single $Sp(V,\Omega)$ orbit, it follows that

$$\langle k_{\Gamma_1},k_{\Gamma_2}\rangle_K = \mathrm{Det}(I-Z_{\Gamma_2}Z_{\Gamma_1}) \tag{1.68}$$

whenever $\Gamma_2 \in Lag_+(V,\Omega)$ and Γ_1 is of type $(0,0)$. Now suppose $\Gamma_1,\Gamma_2 \in Lag_+(V,\Omega)$ and for $0 \le t \le 1$ define $\Gamma_t \in Lag_+(V,\Omega)$ by

$$Z_{\Gamma_t} = tZ_{\Gamma_1} \tag{1.69}$$

(observe that our notation is unambiguous at $t = 0$ and $t = 1$).
If $0 \le t < 1$ then Γ_t is of type $(0,0)$ so from (1.68) we have

$$0 \le t < 1 \Rightarrow \langle k_{\Gamma_t}, k_{\Gamma_2} \rangle_K = \mathrm{Det}(I - Z_{\Gamma_2} Z_{\Gamma_t}) \quad . \tag{1.70}$$

It is readily seen that both $\langle k_{\Gamma_t}, k_{\Gamma_2} \rangle_K$ and $\mathrm{Det}(I - Z_{\Gamma_2} Z_{\Gamma_t})$ are continuous in t for $0 \le t \le 1$; taking the limit of (1.70) as $t \to 1$ yields (1.66).

□

Our next result describes the action of $sp(V,\Omega)_\Gamma^{\mathbb{C}}$ on K^Γ for $\Gamma \in \mathrm{Lag}_+(V,\Omega)$.

Proposition 1.22 : If $\Gamma \in \mathrm{Lag}_+(V,\Omega)$ and $\xi \in sp(V,\Omega)_\Gamma^{\mathbb{C}}$ then

$$\xi \cdot k_\Gamma = (\mathrm{Tr}_\Gamma \xi) k_\Gamma \quad . \tag{1.71}$$

Proof: Let $\xi = \xi_1 + i\xi_2$ with $\xi_1, \xi_2 \in sp(V,\Omega)$ and let $g_1^t = \exp t\xi_1$, $g_2^t = \exp t\xi_2$ be the one-parameter subgroups of $Sp(V,\Omega)$ generated by ξ_1, ξ_2 , so that

$$\xi . k_\Gamma = \frac{d}{dt}\{g_1^t \cdot k_\Gamma + i g_2^t \cdot k_\Gamma\}\Big|_{t=0} \quad .$$

By means of Proposition 1.20 (differentiated) and Proposition 1.14, we deduce

$$\xi \cdot k_\Gamma = (\mathrm{Tr}_\Gamma \xi) k_\Gamma + \frac{d}{dt}\{k_{g_1^t \cdot \Gamma} + i k_{g_2^t \cdot \Gamma}\}\Big|_{t=0} \ .$$

Since $\xi . k_\Gamma$ lies in K^Γ we have

$$\frac{d}{dt}\{k_{g_1^t \cdot \Gamma} + i k_{g_2^t \cdot \Gamma}\}\Big|_{t=0} = \mu k_\Gamma$$

for some $\mu \in \mathbb{C}$. Now

$$\begin{aligned} \mu &= \langle \mu k_\Gamma, k_{\Gamma_0} \rangle_K \\ &= \langle \frac{d}{dt}\{k_{g_1^t \cdot \Gamma} + i k_{g_2^t \cdot \Gamma}\}\Big|_{t=0}, k_{\Gamma_0} \rangle_K \\ &= \frac{d}{dt}\langle k_{g_1^t \cdot \Gamma} + i k_{g_2^t \cdot \Gamma}, k_{\Gamma_0} \rangle_K \Big|_{t=0} \\ &= 0 \end{aligned}$$

in view of (1.62). The result follows.

$\square$

<u>Symplectic Vector Spaces</u> : <u>Normals</u>.

Let L be an isotropic subspace of (V,Ω) . The restriction of Ω to $L^\perp$ has kernel precisely L and so descends through the natural epimorphism $\pi_L : L^\perp \to L^\perp / L$ to yield a symplectic form Ω_L on $L^\perp/L$.

We refer to the symplectic vector space $(L^{\perp}/L,\Omega_L)$ as the symplectic normal space of L .

Each $g \in Sp(V,\Omega;L)$ induces a unique linear automorphism g_L of $L^{\perp}/L$ such that

$$g_L \circ \pi_L = \pi_L \circ g|L^{\perp} \tag{1.72}$$

-moreover this g_L preserves Ω_L . In this way we obtain a natural Lie group epimorphism

$$\rho_L : Sp(V,\Omega;L) \to Sp(L^{\perp}/L,\Omega_L) : g \mapsto g_L \ . \tag{1.73}$$

Structures on (V,Ω) (such as Hilbert structures, polarizations) induce corresponding structures on symplectic normals (subject to compatibility requirements in some cases). We say that such induced structures are obtained by passing to the symplectic normal.

Thus, let J be a Hilbert structure for (V,Ω) . The isotropic nature of L is equivalent to the condition that L and JL be orthogonal with respect to (1.7). $L \oplus JL$ is then a complex subspace of V^J , whose Hermitian orthocomplement therefore coincides with the (Ω-) orthocomplement $L^{\perp} \cap (JL)^{\perp}$, which we denote by V_L . We have direct sum decompositions

$$L^{\perp} = L \oplus V_L \tag{1.74}$$

$$V = L \oplus V_L \oplus JL \tag{1.75}$$

which are orthogonal with respect to (1.7). Orthogonal projection of V on V_L relative to (1.75) is denoted

$$P_L : V \to V_L \, . \tag{1.76}$$

The natural epimorphism $\pi_L : L^{\perp} \to L^{\perp}/L$ restricts to give an isomorphism

$$\check{\pi}_L : (V_L, \Omega|V_L) \to (L^{\perp}/L, \Omega_L) \tag{1.77}$$

of symplectic vector spaces. We transport $J|V_L$ via $\check{\pi}_L$ to obtain a Hilbert structure J_L for $(L^{\perp}/L, \Omega_L)$. In the sequel we shall frequently identify $(V_L, \Omega|V_L; J|V_L)$ and $(L^{\perp}/L, \Omega_L; J_L)$ by means of $\check{\pi}_L$ for reasons of convenience.

Let Γ be a polarization of (V,Ω) such that $L^{\mathbb{C}} \subset \Gamma$. Then $\Gamma_L = \pi_L^{\mathbb{C}}\Gamma$ is naturally a polarization of $(L^{\perp}/L, \Omega_L)$. Moreover, the assignment of Γ_L to Γ is a bijection to $\mathrm{Lag}((L^{\perp}/L)^{\mathbb{C}}, \Omega_L^{\mathbb{C}})$ from the set of polarizations of (V,Ω) which contain $L^{\mathbb{C}}$. The types of Γ and Γ_L are related by

$$(r(\Gamma_L), i(\Gamma_L)) = (r(\Gamma) - \dim L, i(\Gamma)) \ . \tag{1.78}$$

In particular, Γ_L is positive iff Γ is positive.

We define $\Gamma^L \in \mathrm{Lag}_+(V,\Omega)$ by

$$\Gamma^L = L^{\mathbb{C}} + \Gamma_0 \cap V_L^{\mathbb{C}} \quad . \tag{1.79}$$

Γ^L has real part $L^{\mathbb{C}}$, and the polarization $(\Gamma^L)_L$ of $(L^\perp/L,\Omega_L)$ arises from J_L as Γ_0 arises from J :

$$(\Gamma^L)_L = \mathrm{im}(J_L)_+ \quad . \tag{1.80}$$

It is routine to check that Z_{Γ^L} (which we shall for brevity also denote by Z_L) is given by

$$v_1,v_2 \in L \Rightarrow Z_L(v_1 + Jv_2) = v_1 - Jv_2 \tag{1.81}$$

$$v \in V_L \Rightarrow Z_L(v) = 0 \quad . \tag{1.82}$$

Let $\Gamma \in \mathrm{Lag}_+(V,\Omega)$ be such that $L^{\mathbb{C}} \subset \Gamma$. Modulo our identification $\check{\pi}_L$ (1.77) it is clear that

$$Z_{\Gamma_L} = Z_\Gamma | V_L \tag{1.83}$$

and that

$$Z_\Gamma = Z_L + Z_{\Gamma_L} \circ P_L \quad . \tag{1.84}$$

<u>Proposition 1.23</u> : Let $g \in Sp(V,\Omega;L)$. If Γ is a positive

polarization of (V,Ω) such that $L^{\mathbb{C}} \subset \Gamma$ then

$$\mathrm{Det}\{C_g(I+Z_gZ_\Gamma)\} = \mathrm{Det}\{C_{g_L}(I+Z_{g_L}Z_{\Gamma_L})\}\cdot\mathrm{Det}(g|L) \ . \tag{1.85}$$

In particular we have

$$\mathrm{Det}\{C_g(I+Z_gZ_L)\} = \mathrm{Det}C_{g_L}\cdot\mathrm{Det}(g|L) \ . \tag{1.86}$$

Proof: If $v_1, v_2 \in L$ then from (1.81) and (1.84) we deduce

$$C_g(I+Z_gZ_\Gamma)(v_1 + Jv_2) = gv_1 + Jgv_2 \ .$$

Consequently we have

$$\mathrm{Det}\{C_g(I+Z_gZ_\Gamma)|L \oplus JL\} = \mathrm{Det}(g|L) \ . \tag{1.87}$$

Similar use of (1.82) and (1.84) yields

$$\mathrm{Det}\{C_g(I+Z_gZ_\Gamma)|V_L\} = \mathrm{Det}\{C_{g_L}(I+Z_{g_L}Z_{\Gamma_L})\}. \tag{1.88}$$

Since V is the $\mathbb{C}$-linear direct sum of $L \oplus JL$ and V_L, (1.85) follows at once from (1.87) and (1.88). Since (1.80) implies $Z_{(\Gamma^L)_L} = 0$, setting $\Gamma = \Gamma^L$ in (1.85) gives (1.86).

$\square$

Remark 1.24 : We can pass canonical vectors to the symplectic normal.

Choose analogues k_1, k_2 of k_{Γ_0} (see (1.61)) for $(L^\perp/L, \Omega_L)$, $(L \oplus JL, \Omega|L \oplus JL)$; by means of $\check{\pi}_L$ we can view k_1 as an analogue of k_{Γ_0} for $(V_L, \Omega|V_L)$. k_1 and k_2 can be (jointly) normalized by requiring that $k_1 \wedge k_2 = k_{\Gamma_0}$. As a polarization of $(L \oplus JL, \Omega|L \oplus JL)$, L then determines a canonical vector $k_L \in K^L$ (which we can view an element of $\Lambda^r L^{\mathbb{C}}$, for $r = \dim L$) . If Γ is a positive polarization of (V, Ω) such that $L^{\mathbb{C}} \subset \Gamma$, then $\Gamma_L \in \mathrm{Lag}_+(L^\perp/L, \Omega_L)$ determines a canonical vector $k_{\Gamma_L} \in K^{\Gamma_L}$.

//

Vector Bundles and Manifolds.

Let (E, ω) be a real symplectic vector bundle of rank $2m$ over the manifold X ; thus $E \to X$ is a rank $2m$ real vector bundle and ω is a fibrewise nonsingular section of $\Lambda^2 E^*$ over X .

The symplectic frame bundle $Sp(E,\omega)$ of (E,ω) (modelled on (V,Ω)) has as fibre over $x \in X$ the set of all real linear isomorphisms $b: V \to E_x$ satisfying

$$\omega_x(bv_1, bv_2) = \Omega(v_1, v_2) \tag{1.89}$$

for $v_1, v_2 \in V$. $Sp(E,\omega)$ is naturally a principal $Sp(V,\Omega)$ bundle over X with $Sp(V,\Omega)$ action given by composition:

$$b.g = b \circ g \tag{1.90}$$

for $b \in Sp(E,\omega)$ and $g \in Sp(V,\Omega)$. (E,ω) is naturally associated to $Sp(E,\omega)$ via the natural representation of $Sp(V,\Omega)$ on V .

A polarization of (E,ω) is a complex subbundle F of $E^{\mathbb{C}}$ such that $F_x \in Lag(E_x^{\mathbb{C}},\omega_x^{\mathbb{C}})$ for each $x \in X$. If $i(F_x) = 0$ for all $x \in X$, then we say that F is positive. If $(r(F_x),i(F_x))$ is independent of $x \in X$, then we say that F is regular and refer to this common fibre type as the type $(r(F),i(F))$ of F .

Let F be a polarization of (E,ω) , let $F^0 \subset (E^{\mathbb{C}})^*$ denote the annihilator of F , and define the canonical bundle K^F of F to be the top exterior power $\Lambda^m F^0$ of F^0 . The vector bundle isomorphism

$$E^{\mathbb{C}} \to (E^{\mathbb{C}})^* : v \mapsto v \lrcorner\, \omega^{\mathbb{C}} \tag{1.91}$$

restricts to an isomorphism from F to F^0 and so induces an isomorphism from $\Lambda^m F$ to K^F .

The complex line bundle $\Lambda^{2m}(E^{\mathbb{C}})^*$ is canonically trivialized by the Liouville (volume) form

$$\lambda = (-1)^{\frac{m(m-1)}{2}} \frac{\omega^m}{m!} . \tag{1.92}$$

Denote by $\underline{\mathbb{C}}$ the product complex line bundle $X \times \mathbb{C} \to X$. A pseudo-Hermitian structure

$$\langle\cdot,\cdot\rangle_K : \Lambda^m(E^{\mathbb{C}})^* \times \Lambda^m(E^{\mathbb{C}})^* \to \underline{\mathbb{C}} \tag{1.93}$$

is defined by

$$\langle\alpha,\beta\rangle_K \lambda = i^m(\alpha\wedge\bar{\beta}) \tag{1.94}$$

for $\alpha,\beta \in \Lambda^m(E_x^{\mathbb{C}})^*$, $x \in X$.

We say that the pair (F,G) of positive polarizations of (E,ω) is transverse iff $F \cap \bar{G} = 0$. In this case $\langle\cdot,\cdot\rangle_K$ restricts to a nonsignular sesquilinear pairing

$$\langle\cdot,\cdot\rangle_K : K^F \times K^G \to \underline{\mathbb{C}} \quad . \tag{1.95}$$

As can be seen from the existence of the pairing (1.95) for a fixed choice of type $(0,0)$ polarization G , the Chern class $c[K^F]$ of the complex line bundle K^F is independent of the choice of positive polarization F of (E,ω) ; we may thus define the (first) Chern class $c_1(E,\omega)$ of (E,ω) by

$$c_1(E,\omega) = c[K^F] \in H^2(X;\mathbb{Z}) \quad . \tag{1.96}$$

Let F be a regular polarization of (E,ω) and let Γ be a polarization of (V,Ω) having the type of F . The pair (F,Γ) determines a reduction

$$Sp(E,\omega;F) = \{b \in Sp(E,\omega) \mid b^{\mathbb{C}}\Gamma = F\} \tag{1.97}$$

of the symplectic frame bundle of (E,ω) to structure group $Sp(V,\Omega;\Gamma)$. $\Lambda^m F$ (and so K^F) is naturally associated to $Sp(E,\omega;F)$ via the character Det_Γ .

Let D be a subbundle of E . We define the $(\omega\text{-})$ orthocomplement $D^\perp$ of D to be the subbundle of E having fibre $(D^\perp)_x = (D_x)^\perp$ over $x \in X$. We say that D is symplectic, isotropic, coisotropic, or Lagrangian according as $D \cap D^\perp = 0$, $D \subset D^\perp$, $D \supset D^\perp$, or $D = D^\perp$.

Let D be an isotropic subbundle of (E,ω) . The symplectic normal bundle of D is the symplectic vector bundle $(D^\perp/D,\omega_D)$ over X with fibre $(D^\perp/D)_x = D_x^\perp/D_x$ and symplectic form $(\omega_D)_x = (\omega_x)_{D_x}$ over $x \in X$. Let L be an isotropic subspace of (V,Ω) such that $\dim L = \operatorname{rank} D$. The pair (D,L) determines a reduction

$$Sp(E,\omega;D) = \{b \in Sp(E,\omega) \mid bL = D\} \tag{1.98}$$

of $Sp(E,\omega)$ to structure group $Sp(V,\Omega;L)$. Modelling $(D^\perp/D,\omega_D)$ on $(L^\perp/L,\Omega_L)$, $Sp(D^\perp/D,\omega_D)$ is naturally associated to $Sp(E,\omega;D)$ via ρ_L (1.73).

If F is a polarization of (E,ω) such that $D^{\mathbb{C}} \subset F$ then $F_D = F/D^{\mathbb{C}}$ is a polarization of $(D^\perp/D,\omega_D)$; moreover, F_D is positive (regular) iff F is positive (regular). For positive such F with $r = \operatorname{rank} D$ we have

Proposition 1.25 : There exists a canonical isomorphism of complex line bundles

$$K^F \to K^{F_D} \otimes \Lambda^r D^{\mathbb{C}} \quad . \tag{1.99}$$

Proof: If D is Lagrangian then (1.99) is clear when $K^{F_D} = \underline{\mathbb{C}}$. Otherwise, view all bundles in (1.99) as associated to $Sp(E,\omega;D)$. Recalling Remark 1.24 and omitting dependence on the X-variable, we map

$$p(k_{p^{-1}F}) \mapsto p(k_{p^{-1}F_D}) \otimes p(k_L) \quad . \tag{1.100}$$

That this is dependent only on the $Sp(V,\Omega;L)$-orbit of $p \in Sp(E,\omega;D)$ is clear from Propositions 1.20 and 1.23.

□

Remark 1.26 : (1.99) is not strictly canonical, but depends only on choices pertaining to the model L for D . We shall encounter this phenomenon again in the sequel, and shall call canonical any construction which is canonical modulo choices of model.

//

Before stating our next proposition we should make some comments on densities. Let B be a (real or complex) vector bundle with frame bundle $G\ell(B)$ relative to some model. If $\alpha \in \mathbb{R}$ then the (complex) line bundle of α-densities on B is associated to $G\ell(B)$ via the character $|\mathrm{Det}|^{-\alpha}$ of the general linear group; we write

$$\mathcal{D}^{\alpha}(B) = G\ell(B)(|\mathrm{Det}|^{-\alpha}) \tag{1.101}$$

for this density bundle. For more information on densities see Blattner [Br] and Rawnsley [Ry2].

We say that the pair (F,G) of positive polarizations of (E,ω) is regular iff $F \cap \bar{G}$ is a subbundle of $E^{\mathbb{C}}$. In this case, $F \cap \bar{G} = D^{\mathbb{C}}$ for some isotropic subbundle D of (E,ω), (F_D,G_D) is a transverse pair of positive polarizations of $(D^{\perp}/D,\omega_D)$, and we have the following generalization of (1.95):

<u>Proposition 1.27</u> : There exists a canonical nonsingular sesquilinear pairing

$$K^F \times K^G \to \mathcal{D}^{-2}(D) \quad . \tag{1.102}$$

<u>Proof</u>: Apply Proposition 1.25 to both F and G. Since $F_D \cap \bar{G}_D = 0$ we have the pairing $K^{F_D} \times K^{G_D} \to \underline{\mathbb{C}}$ analogous to (1.95). $\Lambda^r D^{\mathbb{C}}$ naturally self-pairs into $\mathcal{D}^{-2}(D)$. The product of these pairings is (1.102).

□

Finally we make some remarks on symplectic manifolds.

Let (X,ω) be a symplectic manifold of dimension $2m$ - thus,

(TX, ω) is a symplectic vector bundle and ω is closed as a 2-form. We assume X to be connected.

We shall denote by $C(X)$ the $\mathbb{C}$-valued functions on X, by $\mathfrak{X}(X)$ the complex vector fields on X, and by $\Omega^k(X)$ the complex k-forms on X. These (and the spaces and operations to be introduced below) have real analogues related to their complex counterparts by complexification; when we consider real analogues we shall state so explicitly.

The complexification $\omega^{\mathbb{C}} \in \Omega^2(X)$ of ω determines an isomorphism

$$\beta : \mathfrak{X}(X) \to \Omega^1(X) : \xi \mapsto \xi \lrcorner \omega^{\mathbb{C}} \; . \tag{1.103}$$

We define a linear map

$$\xi : C(X) \to \mathfrak{X}(X) \tag{1.104}$$

by requiring that

$$\beta(\xi_\phi) = d\phi \tag{1.105}$$

for $\phi \in C(X)$, and refer to ξ_ϕ as the Hamiltonian vector field generated by ϕ.

The Poisson bracket $\{\cdot,\cdot\}$ is defined on $C(X)$ by

$$\phi, \psi \in C(X) \Rightarrow \{\phi, \psi\} = \xi_{\phi}\psi \quad . \tag{1.106}$$

When equipped with this bracket, $C(X)$ becomes a (complex) Lie algebra - the (complex) Poisson algebra of (X,ω) .

We denote by $\mathrm{Ham}_0(X,\omega)$ the sapce of all $\eta \in \mathfrak{X}(X)$ for which $\beta(\eta)$ is closed and by $\mathrm{Ham}(X,\omega)$ the space of all $\xi \in \mathfrak{X}(X)$ for which $\beta(\xi)$ is exact. Under the standard bracket $[\cdot,\cdot]$ of vector fields, $\mathrm{Ham}_0(X,\omega)$ is a subalgebra of $\mathfrak{X}(X)$ and $\mathrm{Ham}(X,\omega)$ is an ideal in $\mathrm{Ham}_0(X,\omega)$.

If we endow $\mathbb{C}$ with the trivial bracket and embed $\mathbb{C}$ in $C(X)$ as the constants, then we have the following central short exact sequence of (complex) Lie algebras:

$$0 \to \mathbb{C} \to C(X) \xrightarrow{\xi} \mathrm{Ham}(X,\omega) \to 0 \quad . \tag{1.107}$$

The preceding material on (X,ω) can be found detailed in Kostant [Kt1].

Since the complex vector field ζ on X lies in $\mathrm{Ham}_0(X,\omega)$ iff the Lie derivative $L_\zeta \omega^{\mathbb{C}}$ vanishes, we have a natural Lie algebra morphism

$$\sim \; : \mathrm{Ham}_0(X,\omega) \to \mathfrak{X}(Sp(TX,\omega)) \tag{1.108}$$

given by the lifting of flows, with the property that $\tilde{\zeta}$ and ζ are related by the bundle projection $\pi: Sp(TX,\omega) \to X$ in the sense that

$$b \in Sp(TX,\omega) \Rightarrow \pi_*(\tilde{\zeta}_b) = \zeta_{\pi(b)} \quad . \tag{1.109}$$

A polarization of (X,ω) is a polarization F of (TX,ω) which is involutive as a subbundle of $TX^{\mathbb{C}}$ - thus

$$\eta_1,\eta_2 \in \Gamma(X;F) \Rightarrow [\eta_1,\eta_2] \in \Gamma(X;F) \tag{1.110}$$

where $\Gamma(X;\cdot)$ denotes sections over X .

Let $U \subset X$ be open. We define

$$C_F(U) = \{\phi \in C(U) \mid \eta \in \Gamma(U;F) \Rightarrow \eta\phi = 0\} \tag{1.111}$$

$$C_F^1(U) = \{\phi \in C(U) \mid \eta \in \Gamma(U;F) \Rightarrow [\xi_\phi,\eta] \in \Gamma(U;F)\} \tag{1.112}$$

$C_F^1(U)$ is a subalgebra of the Poisson algebra $C(U)$ of $(U,\omega|U)$, and $C_F(U)$ is an abelian ideal in $C_F^1(U)$. We denote the associated sheaves by C_F and C_F^1 , and may refer to C_F as the sheaf of (germs of) F-holomorphic functions on X .

Let (F,G) be a pair of positive polarizations of (X,ω) . We say that (F,G) is regular iff $F + \bar{G}$ (equivalently, $F \cap \bar{G}$) is a subbundle of $TX^{\mathbb{C}}$; $F \cap \bar{G}$ is then automatically involutive. We say that (F,G) is strongly regular iff $F + \bar{G}$ is an involutive subbundle of $TX^{\mathbb{C}}$.

The positive polarization F of (X,ω) is strongly regular iff (F,F) is strongly regular. In this case, X is covered by open sets U which carry m-tuples $(\psi_1,\ldots\ldots,\psi_m)$ from $C_F(U)$ such that $(d\psi_1,\ldots\ldots,d\psi_m)$ is a field of bases for F^0 over U (such an m-tuple $(\psi_1,\ldots\ldots,\psi_m)$ is called a local C_F-chart); moreover, if the closed 2-form $\theta \in \Omega^2(X)$ vanishes on F then we may assume each U to carry a 1-form α satisfying

$$\alpha|F = 0 \quad , \quad d\alpha = \theta|U \quad . \tag{1.113}$$

See Rawnsley [Ry1] for details.

§2. The Metaplectic Representation.

The Heisenberg group $N(V,\Omega)$ has underlying manifold $V \times \mathbb{R}$ and has multiplication given by

$$(v_1,t_1)(v_2,t_2) = (v_1+v_2, t_1+t_2-\tfrac{1}{2}\Omega(v_1,v_2)) \tag{2.1}$$

for $v_1,v_2 \in V$ and $t_1,t_2 \in \mathbb{R}$. $N(V,\Omega)$ is a simply-connected connected Lie group with centre $Z = \{0\} \times \mathbb{R}$.

The Heisenberg algebra $n(V,\Omega)$ has underlying vector space $V \oplus \mathbb{R}$ and has bracket given by

$$[v_1 \oplus t_1, v_2 \oplus t_2] = (0 \oplus -\Omega(v_1,v_2)) \tag{2.2}$$

for $v_1,v_2 \in V$ and $t_1,t_2 \in \mathbb{R}$. $n(V,\Omega)$ is a two-step nilpotent Lie algebra with centre $z = \{0\} \oplus \mathbb{R}$.

We may (and shall) naturally identify $n(V,\Omega)$ with the Lie algebra of $N(V,\Omega)$ in such a way that the exponential map becomes the identity on $V \times \mathbb{R}$. The automorphism groups of $N(V,\Omega)$ (as a Lie group) and $n(V,\Omega)$ (as a Lie algebra) are isomorphic - in particular, Aut $N(V,\Omega)$ is a Lie group.

If $g \in Sp(V,\Omega)$ then

$$N_g : N(V,\Omega) \to N(V,\Omega) : (v,t) \mapsto (gv,t) \tag{2.3}$$

is an automorphism of $N(V,\Omega)$; this defines a Lie group monomorphism

$$N:Sp(V,\Omega) \to \mathrm{Aut}\, N(V,\Omega) \quad . \tag{2.4}$$

Remark 2.1 : The full automorphism group of $N(V,\Omega)$ is a semidirect product of V^* by the conformal symplectic group of (V,Ω) .

//

It is a consequence of Schur's lemma that in any irreducible unitary representation π of a Lie group, the centre must act by unitary scalars; π thus restricts to a unitary character of the centre, which we call the central character of π .

The following classical result of Stone & von Neumann is central to our work.

Proposition 2.2 : Corresponding to each nonzero $\lambda \in \mathbb{R}$ there exists (up to unitary equivalence) precisely one irreducible unitary representation of $N(V,\Omega)$ in which the centre Z acts as

$$(0,t) \to e^{i\lambda t}\, I \tag{2.5}$$

for $t \in \mathbb{R}$, where I denotes the identity operator on the relevant (infinite-dimensional complex) Hilbert space.

Proof: See (for example) the account in Rieffel [Rℓ]. □

Remark 2.3 : A representation of $N(V,\Omega)$ as in Proposition 2.2 is near-faithful in the sense that it has discrete central kernel (equal to $\{(0, \frac{2K\pi}{\lambda}) \mid K \in \mathbb{Z}\})$. //

For our purposes it is convenient to consider an irreducible unitary representation

$$W : N(V,\Omega) \to \mathrm{Aut}\,\mathbb{H} \tag{2.6}$$

of $N(V,\Omega)$ on a Hilbert space $\mathbb{H}$ (having inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$) with central character given by

$$t \in \mathbb{R} \Rightarrow W(0,t) = e^{-\frac{1}{i\hbar}t} I \tag{2.7}$$

where $\hbar = h/2\pi$ for some $h \in \mathbb{R}_+$.

We now proceed to introduce the group $Mp^c(V,\Omega)$ and its metaplectic representation.

Proposition 2.4 : If $g \in Sp(V,\Omega)$ then there exists a unitary operator U on $\mathbb{H}$ such that

$$UW(v,t)U^{-1} = W(gv,t) \tag{2.8}$$

for all $(v,t) \in N(V,\Omega)$.

Proof: The irreducible unitary representations $W \circ N_g$ and W have the same central character; the existence of U having the desired properties therefore follows from Proposition 2.2.

□

Remark 2.5 : By Schur's lemma, a unitary U satisfying (2.8) is unique modulo $U(1)$ factors. Since (see Remark 2.3) W is near-faithful, g is uniquely determined by U in (2.8).

//

We denote by $Mp^c(V,\Omega)$ the set of unitary operators U on $\mathbb{H}$ which satisfy (2.8) for some g in $Sp(V,\Omega)$. In view of Remark 2.5 we may define

$$\sigma : Mp^c(V,\Omega) \to Sp(V,\Omega) : U \mapsto g \tag{2.9}$$

when $U \in \mathrm{Aut}\,\mathbb{H}$ and $g \in Sp(V,\Omega)$ satisfy (2.8).

$Mp^c(V,\Omega)$ is a subgroup of $\mathrm{Aut}\,\mathbb{H}$ and σ is a morphism of groups; by Proposition 2.4, σ is surjective; by Remark 2.5, the kernel of σ consists precisely of the unitary scalar multiples of I .

We shall see (in Remark 4.5) that $Mp^c(V,\Omega)$ is a Lie group and that

$$1 \to U(1) \hookrightarrow Mp^c(V,\Omega) \xrightarrow{\sigma} Sp(V,\Omega) \to 1 \tag{2.10}$$

is a central short exact sequence of Lie groups.

Remark 2.6 : (2.10) does not split : there exists no morphism $s:Sp(V,\Omega) \to Mp^c(V,\Omega)$ such that $\sigma\circ s$ is the identity on $Sp(V,\Omega)$. See Proposition 4.11.

//

The representation μ of $Mp^c(V,\Omega)$ on $\mathbb{H}$ coming from inclusion $Mp^c(V,\Omega) \subset \mathrm{Aut}\,\mathbb{H}$ is known as the metaplectic representation. The metaplectic representation is, of course, both faithful and unitary. It is not, however, irreducible; instead, it is the direct sum of two irreducibles. See Sternberg & Wolf [SW].

There is a unique unitary character η of $Mp^c(V,\Omega)$ whose restriction to $U(1) \subset Mp^c(V,\Omega)$ is the squaring map. The kernel of η is a connected double cover of $Sp(V,\Omega)$ which we denote by $Mp(V,\Omega)$ and call the metaplectic group of (V,Ω) . See Propositions 4.7, 4.13, 7.3.

Let $E \subset \mathbb{H}$ be the space of smooth vectors for W ; by definition E consists of those $f \in \mathbb{H}$ for which

$$N(V,\Omega) \to \mathbb{C} : (v,t) \mapsto \langle W(v,t)f,f'\rangle_{\mathbb{H}} \tag{2.11}$$

is a smooth map whenever $f' \in \mathbb{H}$. E is a dense W-stable subspace of $\mathbb{H}$.

The representation W differentiates on E to give a representation

$$\dot{W} : n(V,\Omega) \to \mathrm{End}\, E \tag{2.12}$$

of the Heisenberg algebra, defined by

$$\dot{W}(v \otimes t)f = \frac{d}{ds}\{W(sv,st)f\}\Big|_{s=0} \tag{2.13}$$

and satisfying

$$[\dot{W}(v_1 \otimes t_1),\dot{W}(v_2 \otimes t_2)]f = \frac{1}{i\hbar}\Omega(v_1,v_2)f \tag{2.14}$$

for $v \otimes t$, $v_1 \otimes t_1$, $v_2 \otimes t_2 \in n(V,\Omega)$ and $f \in E$. Thus $\dot{W}$ provides a representation of the (Heisenberg) canonical commutation relations.

By universality, $\dot{W}$ naturally gives rise to a representation (also denoted $\dot{W}$) of the universal enveloping algebra $N(V,\Omega)$ of $n(V,\Omega)$ on E . As u ranges over $N(V,\Omega)$, the seminorms

$$E \to \mathbb{R} : f \mapsto ||\dot{W}(u)f||_{\mathbb{H}} \tag{2.15}$$

endow E with the structure of a Fréchet space. Let E' be the space of all conjugate-linear functionals on E which are continuous in the Fréchet topology, and equip E' with the weak-star topology. The inner product on $\mathbb{H}$ gives us an embedding of $\mathbb{H}$ in E' and we have a rigged Hilbert space

$$E \subset \mathbb{H} \hookrightarrow E' \ . \tag{2.16}$$

The representations W (of $N(V,\Omega)$ on $\mathbb{H}$) and $\dot{W}$ (of $n(V,\Omega)$ on E) admit unique continuous extensions to E' (denoted by the same symbols) which are compatible in the sense that the extension of $\dot{W}$ is the derivative of the extension of W. We further extend $\dot{W}$ by complex linearity to obtain a representation

$$\dot{W}^{\mathbb{C}} : n(V,\Omega)^{\mathbb{C}} \to \text{End } E' \tag{2.17}$$

of the complexified Heisenberg algebra. We shall have more to say on the subject of differentiated representations in §5, §6, and §7.

Some remarks on concrete realizations (or models) of W are in order.

The group $Mp^c(V,\Omega)$ of course depends on W and should therefore more properly be denoted $Mp^c(V,\Omega:W)$. The dependence on W is, however, natural in the following sense.

<u>Remark 2.7</u> : Suppose that for $j = 1,2$

$$W_j : N(V,\Omega) \to \text{Aut } \mathbb{H}_j \tag{2.18}$$

is an irreducible unitary representation with

$$t \in \mathbb{R} \Rightarrow W_j(0,t) = e^{-\frac{1}{i\hbar}t} I \ . \tag{2.19}$$

By the Stone & von Neumann theorem, there exists a unitary operator $S:\mathbb{H}_1 \to \mathbb{H}_2$ which intertwines W_1 and W_2. By Schur's lemma, S is unique modulo $U(1)$. Conjugation

$$Mp^c(V,\Omega:W_1) \to Mp^c(V,\Omega:W_2) : U \mapsto SUS^{-1} \tag{2.20}$$

is consequently an isomorphism of Lie groups, well-defined independently of the choice of S. //

The group $Mp^c(V,\Omega)$ and its metaplectic representation are thus in theory essentially independent of the choice of model for W. In practice, however, there are differences among the models, and one may be more convenient than another.

The standard concrete realization of W is the Schrödinger model, in which $\mathbb{H}$ is the Hilbert space $L^2(\mathbb{R}^m)$, E the Schwartz space $S(\mathbb{R}^m)$ of rapidly-decreasing functions, and E' the space $S'(\mathbb{R}^m)$ of tempered distributions. Unfortunately the metaplectic representation has only been written down explicitly on certain generating subgroups in this model.

In the Bargmann-Segal (BS) model as developed by Rawnsley [Ry5] from ideas of Bargmann [Bn1] [Bn2] and Segal [Sℓ], $\mathbb{H}$, E and E' are all spaces of entire functions on $\mathbb{C}^m$ subject to certain growth conditions, and one can write down the metaplectic representation in explicit form.

For these reasons we shall discuss the metaplectic representation in terms of the BS model. Since it seems to be little known, we consider it appropriate to present an account of the BS model (based on Rawnsley [Ry5]), for which the following section provides the foundation.

§3. Fock spaces and Gaussian integrals.

Recall that (V,Ω) is a $2m$-dimensional real symplectic vector space equipped with a Hilbert structure J, giving rise to the Hermitian inner product

$$\langle v_1,v_2\rangle = \Omega(Jv_1,v_2) + i\Omega(v_1,v_2) \tag{3.1}$$

on the complex vector space V^J. We denote by $|\cdot|$ the norm on V^J coming from $\langle\cdot,\cdot\rangle$.

Let $p \in \mathbb{R}$. We denote by μ_p the (Gaussian) measure on V having density function θ_p, given by

$$\theta_p(z) = h^{-m}(1 + |z|^2)^p \exp(-\frac{|z|^2}{2\hbar}) \tag{3.2}$$

for $z \in V$, relative to the Lebesgue measure on V normalized by $\langle\cdot,\cdot\rangle$.

We denote by $F_p(V,\Omega;J)$ the space of holomorphic $\mathbb{C}$-valued functions on V^J which are square-integrable with respect to μ_p. $F_p(V,\Omega;J)$ is a Hilbert space with inner product $(\cdot,\cdot)_p$ given by

$$(f_1,f_2)_p = \int f_1\bar{f}_2 d\mu_p \tag{3.3}$$

for $f_1, f_2 \in F_p(V,\Omega;J)$. We write $||\cdot||_p$ for the norm induced by $(\cdot,\cdot)_p$. In the special case that p is zero, we write $\mathbb{H}$ in place of $F_0(V,\Omega;J)$.

Remark 3.1 : We extend $(\cdot,\cdot)_0$ to all those pairs f_1, f_2 of holomorphic $\mathbb{C}$-valued functions on V^J for which it makes sense; thus

$$(f_1,f_2)_0 = \int f_1\bar{f}_2 d\mu_0 \tag{3.4}$$

is defined whenever $f_1\bar{f}_2$ is μ_0-integrable. In particular, if $f_1 \in F_{-p}(V,\Omega;J)$ and $f_2 \in F_p(V,\Omega;J)$ for some $p \in \mathbb{R}$ then $(f_1,f_2)_0$ is defined and

$$|(f_1,f_2)_0| \leq ||f_1||_{-p}||f_2||_p \quad . \tag{3.5}$$

//

For $v \in V$ we define $e_v : V \to \mathbb{C}$ by

$$e_v(z) = \exp \frac{1}{2\hbar} \langle z,v \rangle \tag{3.6}$$

for $z \in V$. If $v \in V$ and $p \in \mathbb{R}$ then e_v lies in $F_p(V,\Omega;J)$; indeed

Proposition 3.2 : The linear span of the set $\{e_v | v \in V\}$ is dense in $F_p(V,\Omega;J)$ for every $p \in \mathbb{R}$.

Proof: See Bargmann [Bn2].

□

$\exp \frac{1}{2\hbar} <z,w>$ provides a reproducing kernel for $\mathbb{H} = F_0(V,\Omega;J)$:

Proposition 3.3 : If $f \in \mathbb{H}$ and $z \in V$ then

$$f(z) = (f,e_z)_0 \quad . \tag{3.7}$$

Proof: See Bargmann [Bn1].

□

By virtue of the fact that $\mathbb{H}$ admits a reproducing kernel, all bounded linear operators on $\mathbb{H}$ can be expressed as integral operators:

Proposition 3.4 : Let $A : \mathbb{H} \to \mathbb{H}$ be a bounded linear operator. Then

$$a(z,w) = (Ae_w,e_z)_0 = (Ae_w)(z) \tag{3.8}$$

is an integral kernel for A in the sense that

$$(Af)(z) = \int a(z,w)f(w)d\mu_0(w) \tag{3.9}$$

whenever $f \in \mathbb{H}$ and $z \in V$.

Proof: Let A^* be the $(\cdot,\cdot)_0$-adjoint of A . Then by Proposition 3.3 we have

$$\begin{aligned}(Af)(z) &= (Af,e_z)_0 \\ &= (f,A^* e_z)_0 \\ &= \int f(w)\overline{(A^* e_z)(w)}\,d\mu_0(w) \\ &= \int f(w)\overline{(A^* e_z,e_w)_0}\,d\mu_0(w) \\ &= \int f(w)(Ae_w,e_z)_0\,d\mu_0(w) \quad .\end{aligned}$$

□

<u>Remark 3.5</u> : Observe that $a(z,w)$ is holomorphic in z and anti-holomorphic in w .

//

Define $F \subset \mathbb{H}$ by

$$F = \bigcap_{p \in \mathbb{R}} F_p(V,\Omega;J) \tag{3.10}$$

and topologize F as a countably normed space using $\{||\cdot||_p \mid p \in \mathbb{Z}\}$. The sequence $(f_k)_{k=1}^{\infty}$ in F converges to $f \in F$ iff $||f_k - f||_p \to 0$ as $k \to \infty$ for every $p \in \mathbb{Z}$. Define

$$F' = \bigcup_{p \in \mathbb{R}} F_p(V,\Omega;J) \quad . \tag{3.11}$$

The sequence $(f_k)_{k=1}^{\infty}$ in F' converges (weakly) to $f \in F'$ iff all the f_k $(k \in \mathbb{N})$ belong to some fixed $F_p(V,\Omega;J)$ $(p \in \mathbb{R})$ and $||f_k-f||_p \to 0$ as $k \to \infty$. Moreover:

<u>Proposition 3.6</u> : If $\phi:F \to \mathbb{C}$ is a continuous conjugate-linear functional then there exists a unique $f \in F'$ such that

$$h \in F \Rightarrow \phi(h) = (f,h)_0 \quad . \tag{3.12}$$

Conversely, if $f \in F'$ then (3.12) defines a continuous conjugate-linear functional $\phi:F \to \mathbb{C}$.

<u>Proof</u>: See Bargmann [Bn2].

□

For more detail on these Fock spaces $F_p(V,\Omega;J)$ (along with many other topics) we refer to Bargmann's rather thorough accounts [Bn1], [Bn2].

In our discussion of the Bargmann-Segal model we shall need to evaluate certain Gaussian integrals, to which we turn next.

Let $\Lambda = \Lambda(V,\Omega;J)$ denote the set of all θ in $G\ell(V^J)$ for which

$$v \in V\backslash\{0\} \Rightarrow \mathrm{Re}\langle\theta v,v\rangle > 0 \quad . \tag{3.13}$$

Referring to Blattner [Br], Λ is a contractible open subset of $G\ell(V^J)$ containing the identity; consequently, there is a unique continuous function $\mathrm{Det}^{\frac{1}{2}} : \Lambda(V,\Omega;J) \to \mathbb{C}$ such that

$$\theta \in \Lambda(V,\Omega;J) \Rightarrow (\mathrm{Det}^{\frac{1}{2}}\theta)^2 = \mathrm{Det}\theta$$
$$\& \ \ \mathrm{Det}^{\frac{1}{2}} I = 1 \ . \tag{3.14}$$

<u>Proposition 3.7</u> : Let Z_1 and Z_2 lie in $\bar{\mathcal{D}}$

(i) If either Z_1 or Z_2 lies in $\mathcal{D}$ then $(I - Z_2Z_1) \in G\ell(V^J)$

(ii) If $(I - Z_2Z_1)$ lies in $G\ell(V^J)$ then $(I - Z_2Z_1) \in \Lambda$.

<u>Proof</u>: See Rawnsley [Ry5]. For (i) see also Proposition 1.21 in conjunction with Propositions 1.8, 1.11, 1.17 (ii), noting that $\Gamma \in \mathrm{Lag}_+(V,\Omega)$ is of type $(0,0)$ iff $Z_\Gamma \in \mathcal{D}$.

□

For $Z_1, Z_2 \in \bar{\mathcal{D}}$ and $a_1, a_2, w \in V$ let

$$\phi(Z_1,Z_2;a_1,a_2;w) = \langle w,Z_1w\rangle + \langle Z_2w,w\rangle + 2\langle a_1,w\rangle + 2\langle w,a_2\rangle \ . \tag{3.15}$$

For $Z_1, Z_2 \in \bar{\mathcal{D}}$ such that $(I - Z_2Z_1) \in G\ell(V^J)$ and for $a_1, a_2 \in V$

we define

$$\psi(Z_1,Z_2;a_1,a_2) = 2\langle(I-Z_2Z_1)^{-1}a_1,a_2\rangle + \langle(I-Z_2Z_1)^{-1}a_1,Z_1a_1\rangle + \langle(I-Z_2Z_1)^{-1}Z_2a_2,a_2\rangle \quad . \tag{3.16}$$

<u>Proposition 3.8</u> : Let $Z_1,Z_2 \in \bar{\mathcal{D}}$ and $a_1,a_2 \in V$. If $(I-Z_2Z_1) \in G\ell(V^J)$ then

$$I(Z_1,Z_2;a_1,a_2) = \int \exp \frac{1}{4\hbar}\, \phi(Z_1,Z_2;a_1,a_2;w)d\mu_0(w) \tag{3.17}$$

exists and we have the formula

$$I(Z_1,Z_2;a_1,a_2) = \mathrm{Det}^{\frac{1}{2}}(I-Z_2Z_1)^{-1}\exp \frac{1}{4\hbar}\, \psi(Z_1,Z_2;a_1,a_2) \quad . \tag{3.18}$$

<u>Proof</u>: See Rawnsley [Ry5].

□

This evaluation of a Gaussian integral will suffice for our discussion of the metaplectic representation itself; when we come to consider the differentiated form we shall require:

<u>Proposition 3.9</u> : Let $v,a_1,a_2 \in V$, let $\xi \in sp(V,\Omega)$, and let $Z,Z_1,Z_2 \in \bar{\mathcal{D}}$ with $(I-Z_2Z_1) \in G\ell(V^J)$.

$$\text{(i)} \quad \int \langle v,w\rangle \exp \tfrac{1}{4\hbar}\, \phi(Z_1,Z_2;a_1,a_2;w)\,d\mu_0(w) = \langle (I-Z_2Z_1)^{-1}v, Z_1a_1 + a_2\rangle I(Z_1,Z_2;a_1,a_2) \tag{3.19}$$

$$\text{(ii)} \quad \int \langle A_\xi w,w\rangle \exp \tfrac{1}{4\hbar}\, \phi(Z,0;a_1,a_2;w)\,d\mu_0(w) = \{2\hbar \operatorname{Tr} A_\xi Z + 2\langle A_\xi Za_1,a_2\rangle + \langle A_\xi Za_1,Za_1\rangle + \langle A_\xi a_2,a_2\rangle\} I(Z,0;a_1,a_2) \tag{3.20}$$

Proof: Both parts come from Proposition 3.8 by differentiation (under the integral). In (i) we differentiate with respect to a_1; in (ii) we differentiate $I(Z,Z_{g_t};a_1,a_2)$ along the one-parameter subgroup $g_t = \exp t\xi$. We omit details, merely recording the formula

$$\frac{d}{dt}\{\mathrm{Det}^{\frac{1}{2}}(I-Z_{g_t}Z)^{-1}\}\Big|_{t=0} = \tfrac{1}{2}\operatorname{Tr} A_\xi Z \tag{3.21}$$

(See Proposition 1.6 (ii)).

□

Remark 3.10: **We obtain further formulae of interest and value by repeated differentiation of the formulae in Proposition 3.8.**

//

§4. The Bargmann-Segal Model.

The Bargmann-Segal (BS) representation

$$W : N(V,\Omega) \to \operatorname{Aut} \mathbb{H} \tag{4.1}$$

is defined on $\mathbb{H} = F_0(V,\Omega;J)$ by

$$(W(v,t)f)(z) = \exp\{-\frac{1}{i\hbar}t + \frac{1}{4\hbar}\langle 2z-v,v\rangle\}f(z-v) \tag{4.2}$$

for $(v,t) \in N(V,\Omega)$, $f \in \mathbb{H}$, and $z \in V$. W is an irreducible unitary representation (see Kobayashi [Ki]) which satisfies the condition (2.7).

The space $\mathcal{E}$ of smooth vectors for W is precisely the space $\mathcal{F}$ (3.10):

$$\mathcal{E} = \bigcap_{p \in \mathbb{R}} F_p(V,\Omega;J) \quad . \tag{4.3}$$

Moreover the topology on $\mathcal{F}$ coincides with that on $\mathcal{E} = \mathcal{F}$ defined by the seminorms (2.15). By virtue of the duality in Proposition 3.6 we may identify the (conjugate-linear weak-star) dual $\mathcal{E}'$ of $\mathcal{E}$ with the space $\mathcal{F}'$ (3.11):

$$\mathcal{E}' = \bigcup_{p \in \mathbb{R}} F_p(V,\Omega;J) \ . \tag{4.4}$$

The differentiated representation $\dot{W}$ is given by the formula

$$(\dot{W}(v\otimes t)f)(z) = -\frac{1}{i\hbar}tf(z) + \frac{1}{2\hbar}<z,v>f(z) - df_z(v) \qquad (4.5)$$

for $v\otimes t \in n(V,\Omega)$, $f \in E$, and $z \in V$.

Remark 4.1 : Of course, (4.5) makes sense for any holomorphic function $f:V^J \to \mathbb{C}$ and indeed defines the extension

$$\dot{W} : n(V,\Omega) \to \mathrm{End}\, E' \quad . \qquad (4.6)$$

However, of the spaces $E, F_p(V,\Omega;J)$, E' , only E and E' are $\dot{W}$-stable.

//

Having defined the BS model, we can consider the concomitant metaplectic representation. Recall from §2 that $Mp^c(V,\Omega)$ consists of all those unitary operators U on $\mathbb{H}$ which satisfy

$$(v,t) \in N(V,\Omega) \Rightarrow W(gv,t) = UW(v,t)U^{-1} \qquad (4.7)$$

for some $g \in Sp(V,\Omega)$ and that we write $\sigma(U) = g$ when (4.7) holds.

As bounded linear operators on $\mathbb{H}$, elements of $Mp^c(V,\Omega)$ possess integral kernels (see Proposition 3.4). By means of our parametrization of $Sp(V,\Omega)$ (see §1) and our evaluation of Gaussian integrals (see §3), we can make these integral kernels explicit as follows.

<u>Proposition 4.2</u> : Let $g \in Sp(V,\Omega)$. If $U \in Mp^c(V,\Omega)$ is such that $\sigma(U) = g$ then the integral kernel u of U has the form

$$u(z,w) = \lambda\exp\tfrac{1}{4\hbar}\{2\langle C_g^{-1}z,w\rangle - \langle z,Z_{g^{-1}}z\rangle - \langle Z_g w,w\rangle\} \tag{4.8}$$

for some $\lambda \in \mathbb{C}$ satisfying

$$|\lambda^2 \mathrm{Det}C_g| = 1 . \tag{4.9}$$

Conversely, if $\lambda \in \mathbb{C}$ satisfies (4.9) then (4.8) defines the integral kernel of an element $U \in Mp^c(V,\Omega)$ such that $\sigma(U) = g$.

<u>Proof</u>: We offer an outline, referring to Rawnsley [Ry5] for amplification. Let (4.7) hold. Then, in particular,

$$W(gv,t)Ue_w = UW(v,t)e_w \tag{4.10}$$

for $t \in \mathbb{R}$ and $v,w \in V$. Rearrangement of (4.10) followed by differentiation with respect to v gives the differential equation

$$du_{(z,w)}(gv,v) = \frac{1}{2\hbar}\{\langle z,gv\rangle + \langle v,w\rangle\}u(z,w) \tag{4.11}$$

for $z,w,v \in V$. In view of Remark 3.5, $du_{(z,w)}(y,x)$ is J-linear in y and J-antilinear in x ; taking linear and antilinear parts of (4.11) and rearranging gives

$$\begin{aligned} du_{(z,w)}(v,0) &= \frac{1}{2\hbar}\{\langle C_g^{-1}v,w\rangle - \langle z,Z_{g^{-1}}v\rangle\}u(z,w) \\ du_{(z,w)}(0,v) &= \frac{1}{2\hbar}\{\langle C_g^{-1}z,v\rangle - \langle Z_g v,w\rangle\}u(z,w) \quad . \end{aligned} \tag{4.12}$$

The differential equations (4.12) have solution (4.8) for $\lambda \in \mathbb{C}$. The unitary nature of U forces $||e_0||_0^2 = ||Ue_0||_0^2$; but $||e_0||_0^2 = e_0(0) = 1$ and an application of Proposition 3.8 shows that $||Ue_0||_0^2 = |\lambda^2 \mathrm{Det}\, C_g|$, whence (4.9). The converse is established by reversing the argument.

□

We refer to $(\lambda,g) \in \overset{\circ}{\mathbb{C}} \times Sp(V,\Omega)$ as the parameters of $U \in Mp^c(V,\Omega)$ (in the notation of Proposition 4.2).

Proposition 4.3 : If $U \in Mp^c(V,\Omega)$ has kernel u and parameters (λ,g) then the kernel u^* of U^* is given by

$$u^*(z,w) = \overline{u(w,z)} \tag{4.13}$$

and U^* has parameters $(\bar{\lambda},g^{-1})$.

Proof: $$u^*(z,w) = (U^* e_w, e_z)_0$$
$$= (e_w, Ue_z)_0$$
$$= \overline{(Ue_z, e_w)_0}$$
$$= \overline{u(w,z)} \quad .$$
□

We can now describe the group multiplication (by composition) of $Mp^c(V,\Omega)$ in terms of integral kernels and parameters.

Proposition 4.4 : If $U_1, U_2 \in Mp^c(V,\Omega)$ have kernels u_1, u_2 and parameters (λ_1, g_1), (λ_2, g_2) then the kernel u of U_1U_2 is given by

$$u(z,w) = \int u_1(z,v)u_2(v,w)d\mu_0(v) \tag{4.14}$$

and U_1U_2 has parameters

$$(\lambda_1\lambda_2 \mathrm{Det}^{\frac{1}{2}}(I - Z_{g_1} Z_{g_2^{-1}})^{-1}, g_1g_2) \quad . \tag{4.15}$$

Proof: Using Proposition 4.3 we compute u as follows:

$$u(z,w) = (U_1U_2e_w, e_z)_0$$
$$= (U_2e_w, U_1^* e_z)_0$$

$$= \int (U_2 e_w)(v)\overline{(U_1^* e_z)(v)}\, d\mu_0(v)$$

$$= \int u_2(v,w) u_1(z,v) d\mu_0(v) \quad .$$

Let (λ, g) be the parameters of $U_1 U_2$. Since σ is a group morphism, we see at once that $g = g_1 g_2$; to determine λ , evaluate (4.14) at $z = w = 0$ with the aid of Proposition 3.8.

□

Remark 4.5 : We see now that $Mp^c(V,\Omega)$ is a Lie group (diffeomorphic to $U(1) \times Sp(V,\Omega)$) and that (2.10) is a central short exact sequence of Lie groups.

//

Remark 4.6 : On the strength of Propositions 4.2 and 4.4 we may with impunity regard $Mp^c(V,\Omega)$ as the Lie group with underlying manifold

$$\{(\lambda,g) \in \mathring{\mathbb{C}} \times Sp(V,\Omega) \mid |\lambda^2 \mathrm{Det}\, C_g| = 1\} \tag{4.16}$$

in which the product $(\lambda_1, g_1)\cdot(\lambda_2 g_2)$ is given by (4.15).

//

Proposition 4.7 : A unitary character

$$\eta : Mp^c(V,\Omega) \to U(1) \tag{4.17}$$

is defined by the prescription

$$\eta(U) = \lambda^2 \ \mathrm{Det}\ C_g \tag{4.18}$$

for $U \in Mp^c(V,\Omega)$ with parameters (λ,g) .

η restricts to $U(1) \to Mp^c(V,\Omega)$ as the squaring map.

<u>Proof</u>: A routine computation based on Propositions 1.3 and 4.4 reveals that η is a character. The unitary nature of η follows from Proposition 4.2. That $\eta|U(1)$ is the squaring map is clear. $\square$

We shall see (in Proposition 7.3) that η is the unique unitary character of $Mp^c(V,\Omega)$ whose restriction to $U(1)$ is the squaring map. Recall that we refer to the kernel of η as the metaplectic group $Mp(V,\Omega)$. (2.10) restricts to a central short exact sequence

$$1 \to \mathbb{Z}_2 \hookrightarrow Mp(V,\Omega) \xrightarrow{\sigma} Sp(V,\Omega) \to 1 \tag{4.19}$$

of Lie groups.

We turn now to a consideration of maximal compact groups.

For convenience let us denote the unitary group $U(V,\Omega;J)$ simply by $U(V)$. Denote by $MU^c(V)$ the full preimage of $U(V) \subset Sp(V,\Omega)$ under $\sigma:Mp^c(V,\Omega) \to Sp(V,\Omega)$. $MU^c(V)$ is a maximal compact subgroup of $Mp^c(V,\Omega)$ and (2.10) restricts to a central short exact sequence of Lie groups

$$1 \to U(1) \hookrightarrow MU^c(V) \xrightarrow{\sigma} U(V) \to 1 \quad . \tag{4.20}$$

The action of $MU^c(V)$ in the metaplectic representation is the one we should expect:

Proposition 4.8 : In the metaplectic representation

$$\mu : Mp^c(V,\Omega) \to \mathrm{Aut}\,\mathbb{H} \tag{4.21}$$

the action of $U \in MU^c(V)$ is given by

$$(Uf)(z) = \lambda f(g^{-1}z) \tag{4.22}$$

for $f \in \mathbb{H}$ and $z \in V$ when U has parameters (λ, g) .

Proof: Since $C_g = g$ and $Z_g = 0$, U has kernel

$$u(z,w) = \lambda \exp \frac{1}{2\hbar} \langle g^{-1}z, w\rangle$$

and therefore

$$\begin{aligned}(Uf)(z) &= \lambda\int f(w) \exp \frac{1}{2\hbar} \langle g^{-1}z, w\rangle d\mu_0(w) \\ &= \lambda(f, e_{g^{-1}z})_0 \\ &= \lambda f(g^{-1}z) \quad .\end{aligned}$$

□

As a corollary we have

<u>Proposition 4.9</u> : The taking of parameters gives a Lie group isomorphism

$$MU^c(V) \to U(1) \times U(V) \tag{4.23}$$

and the sequence (4.20) is split by the morphism

$$U(V) \to MU^c(V) : g \mapsto (1,g) \tag{4.24}$$

(given in parametric form).

<u>Proof</u>: Clear from (4.22) of Proposition 4.8.

□

We refer to the kernel of η on $MU^c(V)$ as the metaunitary group $MU(V)$. $MU(V)$ is maximal compact in $Mp(V,\Omega)$ and (2.10) restricts to a central short exact sequence of Lie groups

$$1 \to \mathbb{Z}_2 \hookrightarrow MU(V) \xrightarrow{\sigma} U(V) \to 1 \tag{4.25}$$

In contrast to Proposition 4.9 we have:

<u>Proposition 4.10</u> : The sequence (4.25) does not split when $V \neq 0$.

<u>Proof</u>: Let $s:U(V) \to MU(V)$ be a morphism which splits (4.25); in parameters, write

$$s(g) = (\lambda_g, g)$$

for $g \in U(V)$, thus defining a unitary character $\lambda : U(V) \to U(1)$ which satisfies

$$g \in U(V) \Rightarrow \lambda_g^2 \operatorname{Det} g = 1 \quad .$$

Since the special unitary group

$$SU(V) = \{g \in U(V) \mid \operatorname{Det} g = 1\}$$

is connected and $\lambda_I = 1$, λ restricts to $SU(V)$ with constant value 1. Under the assumption that $V \neq 0$, we may choose $g \in U(V)$ to have $\operatorname{Det} g = -1$ so that $\lambda_g^2 = -1$. Since $\operatorname{Det} g^2 = 1$ we have $g^2 \in SU(V)$ whence $\lambda_{g^2} = 1$. This contradicts the multiplicativity of λ.

□

As a corollary, we justify Remark 2.6:

<u>Proposition 4.11</u> : The sequence (2.10) does not split when $V \neq 0$.

<u>Proof</u>: Suppose $s : Sp(V,\Omega) \to Mp^c(V,\Omega)$ to be a splitting of (2.10). $\eta \circ s$ is then a (unitary) character of the connected semisimple Lie group $Sp(V,\Omega)$, and is therefore trivial. Consequently, s is in fact a splitting of (4.19), and restriction to $U(V)$ gives a splitting of (4.25). This contradicts Proposition 4.10.

□

<u>Remark 4.12</u> : Intermediate to Propositions 4.10 and 4.11 is the nonsplit nature of the sequence (4.19) when $V \neq 0$. //

<u>Proposition 4.13</u> : $Mp(V,\Omega)$ is connected when $V \neq 0$.

<u>Proof</u>: Since $Sp(V,\Omega)$ is connected we need only show that the elements $(\pm 1, I)$ of $\mathbb{Z}_2 = \ker \sigma | Mp(V,\Omega)$ can be joined by a path in $Mp(V,\Omega)$. Choose a J-stable subspace $L \subset V$ of complex dimension one. If $t \in \mathbb{R}$ then

$$g_t = \begin{cases} (\cos 2\pi t)I + (\sin 2\pi t)J & \text{on } L \\ I & \text{on } L^{\perp} \end{cases}$$

lies in $U(V)$ and $C_{g_t} = g_t$ is multiplication by $e^{2\pi it}$ on L and the identity on $L^{\perp}$, so that $\operatorname{Det} C_{g_t} = e^{2\pi it}$. Consequently,

$$g : [0,1] \to U(V) \subset Sp(V,\Omega)$$

is a smooth loop at I and defining

$$\lambda : \mathbb{R} \to U(1) : t \mapsto e^{-\pi it}$$

gives a smooth path

$$(\lambda,g) : [0,1] \to MU(V) \subset Mp(V,\Omega)$$

from $(1,I)$ to $(-1,I)$.

□

<u>Remark 4.14</u> : We could of course have allowed L to have arbitrary odd dimension over $\mathbb{C}$. Had the complex dimension of L been even, then the lift $(\lambda,g) : [0,1] \to MU(V)$ would have been a smooth loop at $(1,I)$.

//

We could continue in this vein, establishing well-known (and not so well-known) properties of the metaplectic representation in the Bargmann-Segal context, but lack of space prohibits an extensive account. It should by now be clear, however, that the BS model is particularly well-suited to dealing with the metaplectic representation. In the coming sections we shall see further evidence in support of this viewpoint.

§5. Vacuum States for Polarizations.

Following Kostant [Kt2] we refer to elements of E' as symplectic spinors for (V,Ω) . In this section we shall be primarily concerned with symplectic spinors annihilated by positive polarizations as follows. Each $\Gamma \in \mathrm{Lag}_+(V,\Omega)$ can be considered as an abelian Lie algebra embedded in $n(V,\Omega)^{\mathbb{C}}$ and so carries a representation on E' coming from $\dot{W}^{\mathbb{C}}$ by restriction. We shall see (in Proposition 5.4) that the Γ-invariant vectors for this representation form a one-dimensional complex subspace of E' .

In the BS model, the representation $\dot{W}^{\mathbb{C}}$ of $n(V,\Omega)^{\mathbb{C}}$ on E' takes the form

$$\begin{aligned}\dot{W}^{\mathbb{C}}((v_1+iv_2) \otimes (t_1+it_2))f(z) &= -\frac{1}{i\hbar}(t_1+it_2)f(z) \\ &\quad + \frac{1}{2\hbar}\langle z,v_1-Jv_2\rangle f(z) - df_z(v_1+Jv_2)\end{aligned} \tag{5.1}$$

for $v_1 \otimes t_1$, $v_2 \otimes t_2 \in n(V,\Omega)$, $f \in E'$, and $z \in V$; see (4.5). Note that df_z is $\mathbb{C}$-linear on V^J in view of the holomorphicity of f . Of course, (5.1) makes sense whenever $f:V^J \to \mathbb{C}$ is holomorphic.

Proposition 5.1 : Let $\Gamma \in \mathrm{Lag}_+(V,\Omega)$. If $f:V^J \to \mathbb{C}$ is holomorphic and satisfies

$$v \in \Gamma \Rightarrow \dot{W}^{\mathbb{C}}(v \otimes 0)f = 0 \tag{5.2}$$

then f is of the form

$$z \in V \Rightarrow f(z) = \lambda \exp \frac{1}{4\hbar} \langle z, Z_\Gamma z \rangle \tag{5.3}$$

for some $\lambda \in \mathbb{C}$.

Proof: By means of the Z-transform (1.49) we see that (5.2) holds iff

$$v \in V \Rightarrow df_z(v) = \frac{1}{2\hbar} \langle z, Z_\Gamma v \rangle f(z) \ . \tag{5.4}$$

The differential equations (5.4) have holomorphic solutions (5.3) for $\lambda \in \mathbb{C}$.

□

If $\Gamma \in \mathrm{Lag}_+(V,\Omega)$ then we define the holomorphic function $f_\Gamma : V^J \to \mathbb{C}$ by

$$z \in V \Rightarrow f_\Gamma(z) = \exp \frac{1}{4\hbar} \langle z, Z_\Gamma z \rangle \ . \tag{5.5}$$

Proposition 5.2 : If $\Gamma \in \mathrm{Lag}_+(V,\Omega)$ then $f_\Gamma \in E'$.

Proof: An elementary analysis based on Fubini's theorem reveals that f_Γ lies in $F_{-r(\Gamma)}(V,\Omega;J) \subset E'$. □

Remark 5.3 : If Γ is of type $(0,0)$ then f actually lies in E .

//

If Γ is a polarization of (V,Ω) then, by virtue of (1.34), Γ is an abelian (complex) Lie algebra embedded in $n(V,\Omega)^{\mathbb{C}}$, thus

$$\Gamma \to n(V,\Omega)^{\mathbb{C}} : v \mapsto v \oplus 0 \quad . \tag{5.6}$$

The composite of (2.17) and (5.6) is a representation of Γ on E' :

$$\dot{W}^{\Gamma} : \Gamma \to \mathrm{End}\, E' \ . \tag{5.7}$$

We now have the result stated at the outset of this section.

<u>Proposition 5.4</u> : If Γ is a positive polarization of (V,Ω) then the subspace $(E')^{\Gamma}$ of E' annihilated by $\dot{W}^{\Gamma}$ is the complex line spanned by f_{Γ} .

<u>Proof</u>: Immediate from Propositions 5.1 and 5.2.

□

If $\Gamma \in \mathrm{Lag}_{+}(V,\Omega)$ then we refer to the space $(E')^{\Gamma}$ of symplectic spinors annihilated by Γ as the vacuum state for Γ.

If the polarization Γ of (V,Ω) is not positive then the invariant subspace $(E')^{\Gamma}$ for $\dot{W}^{\Gamma}$ is zero. Passing to the Lie algebra cohomology $H^{*}(\Gamma;E')$ of Γ with values in E' (regarded as a Γ-module via $\dot{W}^{\Gamma}$), Blattner & Rawnsley [BR] have established the following.

<u>Proposition 5.5</u> : Let $\Gamma \in \mathrm{Lag}(V^{\mathbb{C}},\Omega^{\mathbb{C}})$ be an arbitrary polarization of (V,Ω) . $H^p(\Gamma;E')$ is a complex line iff $p = i(\Gamma)$ and is zero otherwise.

<u>Proof</u>: See Blattner & Rawnsley [BR].

□

Throughout this work we shall mainly be interested in positive polarizations (though in passing we may remark on the general case).

The metaplectic action of $Mp^c(V,\Omega)$ on the symplectic spinors annihilated by positive polarizations is readily determined as follows (compare Proposition 1.20).

<u>Proposition 5.6</u> : If Γ lies in $\mathrm{Lag}_+(V,\Omega)$ and $U \in Mp^c(V,\Omega)$ has parameters (λ,g) then

$$Uf_{\Gamma} = \lambda\, \mathrm{Det}^{\frac{1}{2}}(I+Z_gZ_{\Gamma})^{-1} f_{g\Gamma} \quad . \tag{5.8}$$

<u>Proof</u>: Differentiation of (4.7) with respect to the $N(V,\Omega)$ variable yields

$$U\dot{W}(v \oplus t) = \dot{W}(gv \oplus t)U$$

from which it follows (by complexification) that $Uf_\Gamma \in (E')^{g\Gamma}$, so that (by Proposition 5.4) $Uf_\Gamma = \mu f_{g\Gamma}$ for some $\mu \in \mathbb{C}$. To find μ , evaluate at zero:

$$\begin{aligned}\mu &= (Uf_\Gamma)(0)\\ &= (f_\Gamma, U^* e_0)_0\\ &= \lambda \operatorname{Det}^{\frac{1}{2}}(I+Z_g Z_\Gamma)^{-1}\end{aligned}$$

using Propositions 3.8 and 4.3 (and the clear extension of Proposition 3.3 to the context of Remark 3.1).

□

If Γ is a polarization of (V,Ω) then (referring to (1.40)) we denote by $Mp^c(V,\Omega;\Gamma)$ the full preimage of $Sp(V,\Omega;\Gamma) \subset Sp(V,\Omega)$ under $\sigma: Mp^c(V,\Omega) \to Sp(V,\Omega)$.

If $\Gamma \in Lag_+(V,\Omega)$ then (in view of Proposition 5.6) the complex line $(E')^\Gamma$ is mapped to itself under the metaplectic action of $Mp^c(V,\Omega;\Gamma)$, defining a character

$$\tau_\Gamma : Mp^c(V,\Omega;\Gamma) \to \mathring{\mathbb{C}} \tag{5.9}$$

given by

$$\tau_\Gamma(U) = \lambda \operatorname{Det}^{\frac{1}{2}}(I+Z_g Z_\Gamma)^{-1} \tag{5.10}$$

when $U \in Mp^c(V,\Omega;\Gamma)$ has parameters (λ,g) . $Mp^c(V,\Omega;\Gamma)$ supports other characters: among them being $\eta_\Gamma = \eta | Mp^c(V,\Omega;\Gamma)$ and $Det_\Gamma \circ \sigma$.

Proposition 5.7 : If $\Gamma \in Lag_+(V,\Omega)$ then

$$(\tau_\Gamma)^2 \cdot Det_\Gamma \circ \sigma = \eta_\Gamma \quad . \tag{5.11}$$

Proof: Let $U \in Mp^c(V,\Omega;\Gamma)$ have parameters (λ,g) . Then we have

$$\eta(U) = \{\lambda \; Det^{\frac{1}{2}}(I+Z_gZ_\Gamma)^{-1}\}^2\{Det(C_g(I+Z_gZ_\Gamma))\} \quad . \tag{5.12}$$

(5.11) now follows from (1.53) (5.10) and (5.12).

□

Remark 5.8 : If Γ is an arbitrary polarization of (V,Ω) then $Mp^c(V,\Omega;\Gamma)$ acts on the complex line $H^{i(\Gamma)}(\Gamma;E')$ and the character τ_Γ of $Mp^c(V,\Omega;\Gamma)$ so defined continues to satisfy (5.11). See Blattner & Rawnsley [BR] for details.

//

The following result should be compared with Propositions 1.17(ii) and 1.21 :

Proposition 5.9 : If (Γ_1,Γ_2) is a transverse pair of positive polarizations of (V,Ω) then $(\cdot,\cdot)_0$ restricts to define a nonsingular sesquilinear pairing

$$(E')^{\Gamma_1} \times (E')^{\Gamma_2} \to \mathbb{C} \quad \text{and}$$

$$(f_{\Gamma_1}, f_{\Gamma_2})_0 = \mathrm{Det}^{\frac{1}{2}}(I - Z_{\Gamma_2} Z_{\Gamma_1})^{-1} \quad . \tag{5.13}$$

Proof: An instance of Proposition 3.8.

□

Remark 5.10 : In order to determine the way in which the pairing (5.13) transforms under the action of $g \in Sp(V,\Omega)$ on $Lag_+(V,\Omega)$, we can either proceed as in Rawnsley [Ry5] or apply Proposition 5.6; we obtain

$$(f_{g\Gamma_1}, f_{g\Gamma_2})_0 = |\mathrm{Det}\ C_g|\mathrm{Det}^{\frac{1}{2}}(I+Z_{\Gamma_2}Z_g)\mathrm{Det}^{\frac{1}{2}}(I+Z_g Z_{\Gamma_1})(f_{\Gamma_1}, f_{\Gamma_2})_0 \ . \tag{5.14}$$

//

If L is a Lagrangian subspace of (V,Ω) then $L^{\mathbb{C}} \in Lag_+(V,\Omega)$ (indeed, $L^{\mathbb{C}}$ has type $(m,0)$; moreover, all type $(m,0)$ polarizations of (V,Ω) arise in this way). The results of this section consequently have implications for L . In the next section we consider the case of an isotropic subspace of (V,Ω) which is not Lagrangian.

§6. Passing to Symplectic Normals.

Let L be an isotropic subspace of (V,Ω) such that $L \neq 0 \neq L^{\perp}/L$.

We denote by $(E')^L$ the space of all vectors in E' which are annihilated by L in the representation $\dot{W}$, thus

$$(E')^L = \{f \in E' \mid \ell \in L \Rightarrow \dot{W}(\ell \oplus 0)f = 0\} \quad . \tag{6.1}$$

This extends the concept of vacuum state from §5. However, as we shall see in Proposition 6.4, $(E')^L$ is infinite-dimensional as $L^{\perp}/L \neq 0$.

Proposition 6.1 : $(E')^L \subset E'$ is stable under $W(v,0)$ for $v \in L^{\perp}$ and $W(v,0)$ acts trivially on $(E')^L$ for $v \in L$. Thus W induces a representation

$$W^L : N(L^{\perp}/L, \Omega_L) \to \mathrm{Aut}(E')^L \quad . \tag{6.2}$$

Proof: For convenience, write $W(v) = W(v,0)$ and $\dot{W}(v) = \dot{W}(v \oplus 0)$ for $v \in V$. If $v_1, v_2 \in V$ then

$$W(v_1)W(v_2)W(v_1)^{-1} = \exp \frac{1}{i\hbar}\, \Omega(v_1,v_2) W(v_2)$$

which upon differentiation yields

$$[W(v_1),\dot{W}(v_2)] = \frac{1}{i\hbar}\,\Omega(v_1,v_2)W(v_1) \quad . \tag{6.3}$$

If $f \in (E')^L$ and $v_1 \in L^{\perp}$ then (6.3) implies

$$v_2 \in L \Rightarrow \dot{W}(v_2)W(v_1)f = 0$$

whence $W(v_1)f \in (E')^L$. This establishes the $L^{\perp}$ -stability of $(E')^L \subset E'$. If $f \in (E')^L$ then integration of the constraint

$$v \in L \Rightarrow \dot{W}(v)f = 0$$

along one-parameter subgroups shows that

$$v \in L \Rightarrow W(v)f = f$$

whence L acts trivially on $(E')^L$. We may thus define (6.2) by

$$W^L(\pi_L\, v,t) = W(v,t)|(E')^L \tag{6.4}$$

for $v \in L^{\perp}$ and $t \in \mathbb{R}$. It is routine to check that W^L is a representation.

□

Let us denote f_{Γ^L} by f_L for convenience; thus

$$z \in V \Rightarrow f_L(z) = \exp \frac{1}{4\hbar} \langle z, Z_L z \rangle \ . \tag{6.5}$$

Since $L^{\mathbb{C}} \subset \Gamma^L$ it is immediate that $f_L \in (E')^L$.

Proposition 6.2 : If $f \in (E')^L$ then

$$f = (f \circ P_L) \cdot f_L \ . \tag{6.6}$$

Proof: If $g : V^J \to \mathbb{C}$ is holomorphic then

$$\dot{W}(v \oplus 0)(g f_L)(z) = - f_L(z) dg_z(v) \tag{6.7}$$

for $v \in L$ and $z \in V$; consequently $g f_L \in (E')^L$ iff $dg_z|L = 0$ for all $z \in V$. Since dg_z is J-linear for all $z \in V$ it follows that

$$g f_L \in (E')^L \Leftrightarrow (z \in V \Rightarrow dg_z|L \oplus JL = 0) \ . \tag{6.8}$$

Now suppose $f \in (E')^L$ and define g by

$$f = g \cdot f_L \ .$$

From (6.8), g equals $h \circ P_L$ for some holomorphic function $h : V_L \to \mathbb{C}$; since $f_L | V_L \equiv 1$, it is clear that $h = f | V_L$.

□

Let $E_L \subset \mathbb{H}_L \subset E_L'$ be the rigged Hilbert space in the BS model for $(L^{\perp}/L,\Omega_L;J_L)$. Recall that we identify $(V_L,\Omega|V_L;J|V_L)$ and $(L^{\perp}/L,\Omega_L;J_L)$ by means of $\check{\pi}_L$ (1.77).

<u>Proposition 6.3</u> : (i) If $f \in (E')^L$ then

$$f|V_L \in E_L' .$$

(ii) If $h \in E_L'$ then

$$(h \circ P_L)\cdot f_L \in (E')^L .$$

<u>Proof</u>: Holomorphicity is clear; we need only check the appropriate growth conditions.

(i) If $p < 0$, $z_1 \in V_L$, $z_2 \in L \oplus JL$, then

$$(1 + |z_1|^2)^p(1 + |z_2|^2)^p \le (1 + |z_1+z_2|^2)^p \quad . \tag{6.9}$$

Applying the Fubini-Tonelli theorem to (6.9) we see that if $f \in F_p(V,\Omega;J)$ for some $p < 0$ then $f|V_L \in F_p(V_L,\Omega|V_L;J|V_L)$ and

$$||f|V_L||_p \le (||f_L||_p)^{-1}||f||_p \; . \tag{6.10}$$

(ii) If $p < 0$, $z_1 \in V_L$, $z_2 \in L \oplus JL$, then

$$(1 + |z_1+z_2|^2)^{2p} \le (1 + |z_1|^2)^p(1 + |z_2|^2)^p \quad . \tag{6.11}$$

Applying the Fubini-Tonelli theorem to (6.11) we see that if $h \in F_p(V_L,\Omega|V_L;J|V_L)$ for some $p <- \dim L$ then $(h\circ P_L)\cdot f_L \in F_{2p}(V,\Omega;J)$ and

$$||(h\circ P_L)\cdot f_L||_{2p} \leq (||f_L||_p)||h||_p \quad . \tag{6.12}$$

□

We now have the following description of $(E')^L$:

<u>Proposition 6.4</u> : A topological linear isomorphism

$$R_L : (E')^L \to E'_L \tag{6.13}$$

is defined by

$$R_L f = f|V_L \tag{6.14}$$

for $f \in (E')^L$.

<u>Proof</u>: That R_L is a well-defined linear isomorphism is clear from Proposition 6.2 and 6.3. That R_L is actually an isomorphism of topological vector spaces is a consequence of the norm estimates (6.10) (6.12) in view of the comments preceding Proposition 3.6.

□

<u>Remark 6.5</u> : R_L intertwines W^L (6.2) and the BS representation

W_L of $N(L^{\perp}/L,\Omega_L)$ on E'_L : explicitly we have

$$(v,t) \in N(L^{\perp}/L,\Omega_L) \Rightarrow R_L W^L(v,t) R_L^{-1} = W_L(v,t) \ . \qquad (6.15)$$

//

We now proceed to employ R_L in constructing a lift of the Lie group epimorphism ρ_L (1.73) to Mp^c level. Denote by $Mp^c(V,\Omega;L)$ the full preimage of $Sp(V,\Omega;L)$ under $\sigma : Mp^c(V,\Omega) \to Sp(V,\Omega)$. Observe that if $U \in Mp^c(V,\Omega;L)$ then the continuous extension of U to E' stabilizes $(E')^L \subset E'$.

Proposition 6.6 : If $U \in Mp^c(V,\Omega;L)$ then the automorphism $|\mathrm{Det}(\sigma U|L)|^{\frac{1}{2}} R_L U R_L^{-1}$ of E'_L is unitary on $\mathbb{H}_L \subset E'_L$.

Proof: We given an outline and refer to Rawnsley [Ry5] for more detail. Let U have parameters (λ,g) and denote by U_g the element of $Mp^c(L^{\perp}/L,\Omega_L)$ having parameters $(|\mathrm{Det}\ C_{g_L}|^{-\frac{1}{2}},g_L)$. $R_L U R_L^{-1} U_g^{-1}$ is a topological linear automorphism of E'_L which (in view of (4.7) and (6.15)) commutes with W_L . It is clear (for instance, by considering the integral kernel) that

$$R_L U R_L^{-1} = \mu U_g \qquad (6.16)$$

for some $\mu \in \mathbb{C}$. By applying (6.16) to e_0 and evaluating at zero we find

$$\mu = \lambda \ \mathrm{Det}^{\frac{1}{2}}(I + Z_g Z_L)^{-1} |\mathrm{Det}\ C_{g_L}|^{\frac{1}{2}} \tag{6.17}$$

which upon squaring yields

$$\mu^2 = \lambda^2 \ \mathrm{Det}\ C_g \ \mathrm{Det}\{C_g(I + Z_g Z_L)\}^{-1} |\mathrm{Det}\ C_{g_L}| \ . \tag{6.18}$$

Substitution of (1.86) in (6.18) followed by the taking of absolute values results in

$$|\mu|^2 = |\mathrm{Det}(g|L)|^{-1} \ . \tag{6.19}$$

Hence $|\mathrm{Det}(g|L)|^{\frac{1}{2}} R_L U R_L^{-1}$ is unitary on $\mathbb{H}_L$.

□

We therefore define

$$\hat{\rho}_L : Mp^c(V,\Omega;L) \to Mp^c(L^{\perp}/L,\Omega_L) \tag{6.20}$$

by

$$\hat{\rho}_L(U) = |\mathrm{Det}(\sigma U|L)|^{\frac{1}{2}} R_L U R_L^{-1} \tag{6.21}$$

for $U \in Mp^c(V,\Omega;L)$.

<u>Proposition 6.7</u> : $\hat{\rho}_L$ is a Lie group epimorphism which lifts ρ_L :

$$\sigma \circ \hat{\rho}_L = \rho_L \circ \sigma \tag{6.22}$$

and satisfies

$$\eta(\hat{\rho}_L(U)) = \eta(U)\cdot \text{sign Det}(\sigma U|L) \tag{6.23}$$

$$\hat{\rho}_L(zU) = z\hat{\rho}_L(U) \tag{6.24}$$

for $U \in Mp^c(V,\Omega;L)$ and $z \in U(1)$. If $U \in Mp^c(V,\Omega;L)$ has parameters (λ,g) then $\hat{\rho}_L(U)$ has parameters $(\lambda\ \mathrm{Det}^{\frac{1}{2}}(I+Z_gZ_L)^{-1}|\mathrm{Det}(g|L)|^{\frac{1}{2}},\ g_L)$. (6.25)

<u>Proof</u>: That $\hat{\rho}_L$ is a Lie group epimorphism satisfying (6.22) is clear from Proposition 6.6. That $\hat{\rho}_L$ has the stated parametric form is clear from (6.16) (6.17) (6.21). Since $\hat{\rho}_L$ restricts to the identity on $U(1)$ we have (6.24). A straightforward computation based on the parametric form for $\hat{\rho}_L$ (and using (1.86)) yields (6.23).

□

<u>Remark 6.8</u> : If $\phi: Mp^c(V,\Omega;L) \to Mp^c(L^\perp/L,\Omega_L)$ is a Lie group morphism, then ϕ lifts ρ_L iff $\phi = \kappa.\hat{\rho}_L$ for some unitary character κ of $Mp^c(V,\Omega;L)$. In appropriate form, the same remark applies to lifts relative to any central short exact sequence.

//

<u>Remark 6.9</u> : As a particular consequence of Remark 6.8 and elementary connectedness considerations, the Mp^c lifts ϕ of ρ_L which satisfy

the condition

$$\eta\circ\phi = \eta\cdot\text{sign Det}(\sigma(\cdot)|L) \tag{6.26}$$

of Proposition 6.7 are precisely $\hat{\rho}_L$ and $\hat{\rho}_L\cdot\text{sign Det}(\sigma(\cdot)|L)$ (which are distinct iff $L \neq 0$) .

//

Whereas ρ_L always lifts to the level of Mp^c , there is generally no metaplectic lift of ρ_L . Precisely:

<u>Proposition 6.10</u> : If $L \neq 0 \neq L^{\perp}/L$ then there exists no Lie group morphism

$$\psi : Mp(V,\Omega;L) \to Mp(L^{\perp}/L,\Omega_L)$$

which lifts ρ_L .

<u>Proof</u>: Suppose ψ exists - thus, $\sigma\circ\psi = \rho_L\circ\sigma$. ψ must take $(-1,I) \in Mp(V,\Omega;L)$ to $(-1,I) \in Mp(L^{\perp}/L,\Omega_L)$; otherwise (employing the surjectivity of ρ_L) we could split $\sigma : Mp(L^{\perp}/L,\Omega_L) \to Sp(L^{\perp}/L,\Omega_L)$ and so contradict Remark 4.12. ψ therefore extends to an Mp^c lift $\phi : Mp^c(V,\Omega;L) \to Mp^c(L^{\perp}/L,\Omega_L)$ of ρ_L ; ϕ must then satisfy

$$\eta\circ\phi = \eta \ . \tag{6.27}$$

Remark 6.8 tells us that $\phi = \kappa\cdot\hat{\rho}_L$ for some unitary character

κ of $Mp^c(V,\Omega;L)$. From (6.23) and (6.27) we deduce that κ satisfies

$$\kappa^2 \cdot \text{sign Det}(\sigma(\cdot)|L) = 1 \tag{6.28}$$

$Mp^c(V,\Omega;L)$ has two components, $Mp^c(V,\Omega;L)_{+1}$ and $Mp^c(V,\Omega;L)_{-1}$, separated by sign $\text{Det}(\sigma(\cdot)|L)$ - thus, if $U \in Mp^c(V,\Omega;L)$ and $j \in \{+1,-1\}$ then

$$U \in Mp^c(V,\Omega;L)_j \iff \text{sign Det}(\sigma(U)|L) = j \quad .$$

Connectedness of $Mp^c(V,\Omega;L)_{+1}$ clearly forces

$$\kappa|Mp^c(V,\Omega;L)_{+1} = +1 \quad . \tag{6.29}$$

Choose $U \in Mp^c(V,\Omega;L)_{-1}$; U^2 then lies in $Mp^c(V,\Omega;L)_{+1}$. From (6.28) we have $\kappa(U)^2 = -1$ whereas from (6.29) we have $\kappa(U^2) = +1$. This contradicts the multiplicativity of κ .

□

Remark 6.11 : Looking ahead to §8, it is clear that we could base a more devious proof of Proposition 6.10 on Remark 8.13.

//

§7. The Differentiated Metaplectic Representation.

The central short exact sequence (2.10) of Lie groups induces (by differentiation) a central short exact sequence

$$0 \to u(1) \hookrightarrow mp^c(V,\Omega) \xrightarrow{\sigma_*} sp(V,\Omega) \to 0 \tag{7.1}$$

of Lie algebras, where $mp^c(V,\Omega)$ denotes the Lie algebra of $Mp^c(V,\Omega)$. Whereas (2.10) does not split, (7.1) does split; we might say that (2.10) splits at the Lie algebra level.

Proposition 7.1 : The short exact sequence (7.1) splits; indeed, a splitting is provided by the Lie algebra morphism

$$\tfrac{1}{2}\eta_* : mp^c(V,\Omega) \to u(1) \quad . \tag{7.2}$$

Proof: The derivative η_* of the unitary character η is certainly a morphism of Lie algebras; since $u(1)$ is abelian, $\tfrac{1}{2}\eta_*$ is also a Lie algebra morphism. As $\eta|U(1)$ is the squaring map, $\eta_*|u(1)$ is the doubling map; consequently, $\tfrac{1}{2}\eta_*$ splits (7.1).

□

Remark 7.2 : This result enables us to regard $mp^c(V,\Omega)$ as the Lie algebraic direct sum of $u(1)$ and $sp(V,\Omega)$ via the isomorphism

$$mp^c(V,\Omega) \to u(1) \oplus sp(V,\Omega) \tag{7.3}$$

given by

$$x \mapsto (\tfrac{1}{2}\eta_* x) \oplus (\sigma_* x) \tag{7.4}$$

for $x \in mp^c(V,\Omega)$. In terms of this identification η_* and σ_* become

$$\eta_* : mp^c(V,\Omega) \to u(1) : \zeta \oplus \xi \mapsto 2\zeta \tag{7.5}$$

$$\sigma_* : mp^c(V,\Omega) \to sp(V,\Omega) : \zeta \oplus \xi \mapsto \xi \quad . \tag{7.6}$$

//

We are now able to establish the uniqueness of η .

<u>Proposition 7.3</u> : η is the unique unitary character of $Mp^c(V,\Omega)$ whose restriction to $U(1) \hookrightarrow Mp^c(V,\Omega)$ is the squaring map.

<u>Proof</u>: If η' is a unitary character of $Mp^c(V,\Omega)$ such that $\eta'(\lambda) = \lambda^2$ for $\lambda \in U(1)$ then

$$\zeta \in u(1) \Rightarrow \eta'_*(\zeta \oplus 0) = 2\zeta \quad . \tag{7.7}$$

Since $sp(V,\Omega)$ is semisimple we also have

$$\xi \in sp(V,\Omega) \Rightarrow \eta'_*(0 \oplus \xi) = 0 \quad . \tag{7.8}$$

Putting together (7.7) and (7.8) it is clear that $\eta'_* = \eta_*$. The connectedness of $Mp^c(V,\Omega)$ now forces $\eta' = \eta$. □

Remark 7.4 : Observe that if κ is a unitary character of $Mp^c(V,\Omega)$ then $\kappa|U(1)$ must be a power map (being a unitary character of $U(1)$). Since (2.10) does not split, $\kappa|U(1)$ cannot be the identity (thus, $\frac{1}{2}\eta_*$ does not exponentiate to a unitary character of $Mp^c(V,\Omega)$). We now see that η is in fact a generator of the unitary character group of $Mp^c(V,\Omega)$. //

The embedding of $sp(V,\Omega)$ in $mp^c(V,\Omega)$ afforded by the identification (7.3) gives an isomorphism from $sp(V,\Omega)$ to the Lie algebra $mp(V,\Omega)$ of $Mp(V,\Omega)$ whose inverse is the restriction to $mp(V,\Omega)$ of $\sigma_*:mp^c(V,\Omega) \to sp(V,\Omega)$.

Our next result describes the exponential map

$$\exp : mp^c(V,\Omega) \to Mp^c(V,\Omega)$$

in terms of the parametrizations of $Mp^c(V,\Omega)$ (Remark 4.6) and $mp^c(V,\Omega)$ (Remark 7.2).

Proposition 7.5 : Let $\zeta \oplus \xi \in mp^c(V,\Omega)$ and write
$\exp t(\zeta \oplus \xi) = (\lambda_t, g_t) \in Mp^c(V,\Omega)$. Then

$g_t = \exp t\xi \in Sp(V,\Omega)$ and $\lambda_t = z_t \alpha_t$ for $z_t = \exp t\zeta \in U(1)$ and some $\alpha_t \in \overset{\circ}{\mathbb{C}}$ depending smoothly on t and satisfying

$$\alpha_t^2 \, \mathrm{Det}\, C_{g_t} = 1 \quad . \tag{7.9}$$

Moreover

$$\frac{d}{dt}\{\lambda_t\}\Big|_{t=0} = \zeta - \tfrac{1}{2}\,\mathrm{Tr}C_\xi \quad . \tag{7.10}$$

<u>Proof</u>: Since $t(0 \oplus \xi) \in mp(V,\Omega)$ we have

$$\exp t(0 \oplus \xi) = (\alpha_t, g_t) \tag{7.11}$$

for g_t and α_t as stated. Write $z_t = \exp t\zeta$, so that

$$\exp t(\zeta \oplus 0) = (z_t, I) \tag{7.12}$$

and

$$\frac{d}{dt}\{z_t\}\Big|_{t=0} = \zeta \quad . \tag{7.13}$$

Differentiating (7.9) with the aid of Proposition 1.6 (i) yields

$$\frac{d}{dt}\{\alpha_t\}\Big|_{t=0} = -\tfrac{1}{2}\,\mathrm{Tr}C_\xi \quad . \tag{7.14}$$

From (7.11) and (7.12) we obtain

$$(\lambda_t, g_t) = \exp t(\zeta \oplus \xi)$$

$$= \exp t(\zeta \oplus 0) \exp t(0 \oplus \xi)$$

$$= (z_t, I)(\alpha_t, g_t)$$

$$= (z_t \alpha_t, g_t)$$

(by virtue of the centrality of $U(1) \subset Mp^c(V,\Omega)$) . Consequently $\lambda_t = z_t \alpha_t$ which upon differentiation gives (7.10) in view of (7.13) and (7.14). □

We are now at liberty to consider the differentiated metaplectic representation.

The space of smooth vectors for the metaplectic representation $\mu : Mp^c(V,\Omega) \to \mathrm{Aut}\,\mathbb{H}$ is precisely E . The differentiated metaplectic representation

$$\dot{\mu} : mp^c(V,\Omega) \to \mathrm{End}\; E \tag{7.15}$$

is defined by differentiation of μ along one-parameter subgroups - thus

$$(\dot{\mu}(x)f)(z) = \frac{d}{dt}\{\mu(\exp tx)f(z)\}\Big|_{t=0} \tag{7.16}$$

for $x \in mp^c(V,\Omega)$, $f \in E$, and $z \in V$.

Since μ is a representation of $Mp^c(V,\Omega)$ by unitary operators on $\mathbb{H}$, $\dot{\mu}$ represents $mp^c(V,\Omega)$ as a Lie algebra of densely-defined (essentially) skew-adjoint operators in $\mathbb{H}$; in particular, if $x \in mp^c(V,\Omega)$ then

$$(\dot{\mu}(x)f_1,f_2)_0 + (f_1,\dot{\mu}(x)f_2)_0 = 0 \tag{7.17}$$

whenever $f_1,f_2 \in E$.

Arguing along similar lines to those which occur in Proposition 3.4, $\dot{\mu}(x)$ is an integral operator on E for each $x \in mp^c(V,\Omega)$.

<u>Proposition 7.6</u> : If $\zeta \oplus \xi \in mp^c(V,\Omega)$ then the integral kernel $\dot{u}$ of the operator $\dot{\mu}(x)$ is given by the formula

$$\dot{u}(z,w) = \{\zeta-\tfrac{1}{2}TrC_\xi - \frac{1}{4\hbar}(2\langle C_\xi z,w\rangle - \langle z,A_\xi z\rangle + \langle A_\xi w,w\rangle)\}e_w(z) . \tag{7.18}$$

<u>Proof</u>: Let $\exp t(\zeta \oplus \xi) = (\lambda_t,g_t) \in Mp^c(V,\Omega)$ and denote by u_t the integral kernel of $\mu(\lambda_t,g_t)$:

$$u_t(z,w) = \lambda_t \exp\frac{1}{4\hbar}\{2\langle C_{g_t}^{-1}z,w\rangle - \langle z,Z_{g_t^{-1}}z\rangle - \langle Z_{g_t}w,w\rangle\} \tag{7.19}$$

$$(\mu(\lambda_t,g_t)f)(z) = \int u_t(z,w)f(w)d\mu_0(w) \tag{7.20}$$

when $f \in \mathbb{H}$. If $f \in E$ then (7.20) implies

$$(\dot{\mu}(\zeta\otimes\xi)f)(z) = \int \frac{d}{dt} u_t(z,w)\Big|_{t=0} f(w)d\mu_0(w)$$

whence the integral kernel $\dot{u}$ of $\dot{\mu}(\zeta\otimes\xi)$ is

$$\dot{u}(z,w) = \frac{d}{dt}\{u_t(z,w)\}\Big|_{t=0} \quad .$$

This differentiation of (7.19) is performed with the aid of Propositions 1.6 and 7.5 and results in the stated formula (7.18).

□

Remark 7.7 : The differentiated metaplectic representation $\dot{\mu}$ (7.15) extends continuously to E' to give a representation

$$\dot{\mu} : mp^c(V,\Omega) \to \mathrm{End}\, E' \quad . \tag{7.21}$$

If $\zeta\otimes\xi \in mp^c(V,\Omega)$ then $\dot{\mu}(\zeta\otimes\xi)$ is an integral operator on E' with kernel (7.18). $\dot{\mu}$ extends further by complex linearity to a representation

$$\dot{\mu}^{\mathbb{C}} : mp^c(V,\Omega)^{\mathbb{C}} \to \mathrm{End}\, E' \tag{7.22}$$

of the complexification $mp^c(V,\Omega)^{\mathbb{C}}$.

//

The differentiated representations $\dot{W}^{\mathbb{C}}$ (2.17) and $\dot{\mu}^{\mathbb{C}}$ (7.22) are related as follows:

Proposition 7.8 : If $\zeta\otimes\xi \in mp^c(V,\Omega)^{\mathbb{C}}$ and $v \otimes t \in n(V,\Omega)^{\mathbb{C}}$ then

$$[\dot{\mu}^{\mathbb{C}}(\zeta\otimes\xi),\dot{W}^{\mathbb{C}}(v\otimes t)] = \dot{W}^{\mathbb{C}}(\xi v\otimes 0) \ . \tag{7.23}$$

Proof: Differentiate the definitional formula (2.8) along one-parameter subgroups (in both N-variables and Mp^c-variables) and complexify.

□

Let Γ be a positive polarization of (V,Ω). We denote by $mp^c(V,\Omega;\Gamma)$ the Lie algebra of $Mp^c(V,\Omega;\Gamma)$ and by $mp^c(V,\Omega)^{\mathbb{C}}_{\Gamma}$ the full preimage of $sp(V,\Omega)^{\mathbb{C}}_{\Gamma}$ under $\sigma^{\mathbb{C}}_{*} : mp^c(V,\Omega)^{\mathbb{C}} \to sp(V,\Omega)^{\mathbb{C}}$. Observe that $mp^c(V,\Omega;\Gamma)^{\mathbb{C}} \subset mp^c(V,\Omega)^{\mathbb{C}}_{\Gamma}$.

It is apparent from Proposition 7.8 that the (differentiated) metaplectic action of $mp^c(V,\Omega)^{\mathbb{C}}_{\Gamma}$ stabilizes the vacuum state $(E')^{\Gamma} \subset E'$ and so defines a character of the complex Lie algebra $mp^c(V,\Omega)^{\mathbb{C}}_{\Gamma}$. Let us determine this character explicitly.

Define a linear endomorphism M^{Γ} of $sp(V,\Omega)$ by

$$M^{\Gamma}_{\xi} = [C_{\xi},Z_{\Gamma}] + A_{\xi} - Z_{\Gamma}A_{\xi}Z_{\Gamma} \tag{7.24}$$

for $\xi \in sp(V,\Omega)$, and note that

$$\xi \in sp(V,\Omega) \Rightarrow M^{\Gamma}_{\xi}J + JM^{\Gamma}_{\xi} = 0 \ . \tag{7.25}$$

Proposition 7.9 : If $\zeta\otimes\xi \in mp^c(V,\Omega)$ and $\Gamma \in Lag_{+}(V,\Omega)$ then

$$(\dot{\mu}(\zeta\oplus\xi)f_\Gamma)(z) = \{\zeta - \tfrac{1}{2}TrC_\xi - \tfrac{1}{2}TrA_\xi Z_\Gamma + \frac{1}{4\hbar}<z, M_\xi^\Gamma z>\}f_\Gamma(z) \tag{7.26}$$

whenever $z \in V$.

<u>Proof</u>: Evaluating $(\dot{\mu}(\zeta\oplus\xi)f_\Gamma)(z)$ with the aid of Propositions 3.9 and 7.6 yields

$$\begin{aligned}\dot{\mu}(\zeta\oplus\xi)f_\Gamma(z) &= \{\zeta - \tfrac{1}{2}TrC_\xi - \tfrac{1}{2}TrA_\xi Z_\Gamma\}f_\Gamma(z) \\ &+ \frac{1}{4\hbar}\{<z, A_\xi z> - 2<C_\xi z, Z_\Gamma z> - <A_\xi Z_\Gamma z, Z_\Gamma z>\}f_\Gamma(z)\end{aligned} \tag{7.27}$$

(7.26) follows from (7.27) and the observation that $<z,(C_\xi Z_\Gamma + Z_\Gamma C_\xi)z> = 0$.

□

<u>Proposition 7.10</u> : If $(\zeta_1\oplus\xi_1) + i(\zeta_2\oplus\xi_2) \in mp^c(V,\Omega)^{\mathbb{C}}$ and $\Gamma \in Lag_+(V,\Omega)$ then

$$\begin{aligned}&(\dot{\mu}^{\mathbb{C}}((\zeta_1\oplus\xi_1) + i(\zeta_2\oplus\xi_2))f_\Gamma)(z) = \\ &\{(\zeta_1 + i\zeta_2) - \tfrac{1}{2}(TrC_{\xi_1} + TrA_{\xi_1}Z_\Gamma) - \tfrac{1}{2}i(TrC_{\xi_2} + TrA_{\xi_2}Z_\Gamma)\}f_\Gamma(z) \\ &+ \frac{1}{4\hbar}<z, (M_{\xi_1}^\Gamma - JM_{\xi_2}^\Gamma)z>f_\Gamma(z)\end{aligned} \tag{7.28}$$

whenever $z \in V$.

<u>Proof</u>: Complexify (7.26).

□

Our next result gives explicit form to the character of $mp^c(V,\Omega)_\Gamma^{\mathbb{C}}$ determined by its metaplectic action on $(E')^\Gamma \subset E'$.

<u>Proposition 7.11</u> : If $\Gamma \in Lag_+(V,\Omega)$ and $\zeta\oplus\xi \in mp^c(V,\Omega)_\Gamma^{\mathbb{C}}$ then

$$\dot{\mu}^{\mathbb{C}}(\zeta\oplus\xi)f_\Gamma = (\zeta-\tfrac{1}{2}Tr_\Gamma\xi)f_\Gamma \quad . \tag{7.29}$$

<u>Proof</u>: Let $\zeta = \zeta_1 + i\zeta_2$ and $\xi = \xi_1 + i\xi_2$. Routine computations reveal that if $v_1 + iv_2 \in \Gamma$ then

$$(M_{\xi_1}^\Gamma - JM_{\xi_2}^\Gamma)(B_\Gamma(v_1+iv_2)) = \{(\xi_1v_1-\xi_2v_2)-J(\xi_1v_2+\xi_2v_1)\}$$
$$- Z_\Gamma\{(\xi_1v_1-\xi_2v_2) + J(\xi_1v_2+\xi_2v_1)\}$$

whence $M_{\xi_1}^\Gamma - JM_{\xi_2}^\Gamma = 0$. (7.29) therefore follows from (7.28) in view of Proposition 1.14.

□

<u>Remark 7.12</u> : It is interesting (and amusing) to note that we can prove Proposition 7.11 without reference to our general explicit formulae for $\dot{\mu}^{\mathbb{C}}$. We need only adapt the line of proof for Proposition 1.22 to the present situation (noting from Proposition 5.9 that $(f_\Gamma,f_{\Gamma_0})_0 = 1$, differentiating Proposition 5.6 along one-parameter subgroups, and applying Propositions 1.14 and 7.5).

//

§8. Mp^c Structures.

We begin this section with a brief discussion of the general problem of lifting structure groups of principal bundles. See Greub & Petry [GP] and Hess & Krauser [HK] for details.

For a Lie group G, we shall denote by $H^1(X,\underline{G})$ the space of isomorphism classes of principal G bundles over the manifold X. When G is abelian, $H^1(X;\underline{G})$ is naturally Čech cohomology of X with smooth coefficients in G ; in general, $H^1(X;\underline{G})$ has the structure of a pointed set.

Let B be a principal G bundle over X and let

$$1 \to C \hookrightarrow \hat{G} \xrightarrow{\pi} G \to 1 \tag{8.1}$$

be a central short exact sequence of Lie groups.

A π-lifting of B is a principal $\hat{G}$ bundle $\hat{B}$ together with a π-equivariant principal bundle morphism $\hat{B} \to B$. The π-liftings $\hat{B}_1$ and $\hat{B}_2$ are equivalent iff there exists an isomorphism $\hat{B}_1 \to \hat{B}_2$ of principal G bundles which commutes with the respective projections on B. We denote by $L(B;\pi)$ the space of equivalence classes $[\hat{B}]$ of π-lfitings $\hat{B}$ of B.

The sequence (8.1) induces an exact sequence in cohomology, thus:

$$H^1(X;\underline{C}) \to H^1(X;\underline{\hat{G}}) \xrightarrow{\pi} H^1(X;\underline{G}) \xrightarrow{\delta} H^2(X;\underline{C}) \tag{8.2}$$

Proposition 8.1 : (i) B admits π-liftings iff $\delta[B] \in H^2(X;\underline{C})$ is trivial.

(ii) When nonempty, $L(B;\pi)$ is naturally a principal $H^1(X;\underline{C})$ space.

Proof: An exercise in transition functions; see Greub & Petry [GP] or Hess & Krauser [HK].

□

Now suppose (E,ω) to be a real symplectic vector bundle of rank $2m$ over the connected manifold X . We model (E,ω) on the 2m-dimensional real symplectic vector space (V,Ω) . Recall the specific central short exact sequence of Lie groups (2.10):

$$1 \to U(1) \hookrightarrow Mp^c(V,\Omega) \xrightarrow{\sigma} Sp(V,\Omega) \to 1 . \tag{8.3}$$

An Mp^c structure for (E,ω) is a σ-lifting of $Sp(E,\omega)$ - thus a principal $Mp^c(V,\Omega)$ bundle P on X together with a σ-equivariant principal bundle morphism from P to $Sp(E,\omega)$. We shall write $T(E,\omega)$ in place of $L(Sp(E,\omega);\sigma)$.

Remark 8.2 : Observe that an Mp^c structure $P \to Sp(E,\omega)$ is canonically a principal $U(1)$ bundle.

//

In similar fashion we define Mp (or metaplectic) structures for (E,ω) as liftings of $Sp(E,\omega)$ relative to the sequence (4.19) and denote the space of their equivalence classes by $M(E,\omega)$.

We shall have cause to consider maximal compact reductions of structure group in principal bundles. It is well known that these always exist and are all equivalent.

Fix a reduction Q of $Sp(E,\omega)$ to structure group $U(V)$. In canonical fashion, Q converts (E,ω) into a Hermitian complex vector bundle $(E,\omega;Q)$. If P is an Mp^c structure for (E,ω) then the part $P|Q$ of P which lies over $Q \to Sp(E,\omega)$ is a principal $MU^c(V)$ bundle on X equipped with a $(\sigma|MU^c(V))$ - equivariant principal bundle morphism $(P|Q) \to Q$; $P|Q$ is thus an MU^c structure for $(E,\omega;Q)$.

The following rather pleasant result guarantees the unqualified existence of Mp^c structures.

Proposition 8.3 : (E,ω) always admits Mp^c structures.

Proof: Choose a reduction Q of $Sp(E,\omega)$ to structure group $U(V)$. Q extends to an MU^c structure Q^c for $(E,\omega;Q)$ via the splitting (4.24) of (4.20). Q^c extends via the inclusion of $MU^c(V)$ in $Mp^c(V,\Omega)$ to provide an Mp^c structure P for (E,ω) . Note that the MU^c structures $P|Q$ and Q^c are canonically equivalent.

$\square$

Remark 8.4 : Returning to the context of Proposition 8.1 it is clear that if π splits then $L(B;\pi) \neq \emptyset$. Proposition 8.3 illustrates the fact that in order for $L(B;\pi)$ to be nonempty it suffices that π split at the maximal compact level.

//

From Proposition 8.1 we know that $T(E,\omega)$ is a principal $H^1(X;\underline{U(1)})$ space. Let us see this explicitly and independently. Recall that the Chern class gives an isomorphism

$$C : H^1(X;\underline{U(1)}) \to H^2(X;\mathbb{Z}) \quad . \tag{8.4}$$

Let Y be a principal $U(1)$ bundle on X and let P be an Mp^c structure for (E,ω). Form the fibre product

$$Y \dot{\times} F = \{(y,p) \in Y \times P \mid \pi(y) = \pi(p)\} \tag{8.5}$$

of Y and P over X ; $Y \dot{\times} P$ is a principal $U(1) \times Mp^c(V,\Omega)$ bundle over X. Associated to $Y \dot{\times} P$ via the morphism

$$U(1) \times Mp^c(V,\Omega) \to Mp^c(V,\Omega) : (\lambda,U) \mapsto \lambda U \tag{8.6}$$

is an Mp^c structure for (E,ω) (with the obvious projection on $Sp(E,\omega)$) which we denote by P^Y.

This twisting of Mp^c structures by principal $U(1)$ bundles passes to the level of equivalence classes to give a natural action

$$H^1(X;\underline{U(1)}) \times T(E,\omega) \to T(E,\omega):([Y],[P]) \mapsto [P^Y] \tag{8.7}$$

or, via the Chern class (8.4),

$$H^2(X;\mathbb{Z}) \times T(E,\omega) \to T(E,\omega):(c[Y],[P]) \mapsto [P^Y] \quad . \tag{8.8}$$

Proposition 8.5 : $T(E,\omega)$ is naturally a principal $H^2(X;\mathbb{Z})$ space for the action (8.8).

Proof: Fix a $U(V)$-reduction Q of $Sp(E,\omega)$. Let P_1, P_2 be Mp^c structures for (E,ω) and denote by Y_1, Y_2 the principal $U(1)$ bundles associated to $(P_1|Q)$, $(P_2|Q)$ via $\tau_{\Gamma_0} : MU^c(V) \to U(1)$. The Mp^c structures $P_1^{Y_2}$ and $P_2^{Y_1}$ are equivalent; indeed, the MU^c structures $P_1^{Y_2}|Q = (P_1|Q)^{Y_2}$ and $P_2^{Y_1}|Q = (P_2|Q)^{Y_1}$ are equivalent. Consequently,

$$c[Y_2].[P_1] = c[Y_1].[P_2] \tag{8.9}$$

from which the transitivity of (8.8) is clear. Let P be an Mp^c structure and Y a principal $U(1)$ bundle. Denote by L the (Hermitian) complex line bundle associated to Y via $U(1) \to G\ell(\mathbb{C})$. An equivalence of Mp^c structures $P^Y \to P$ forces an isomorphism of (Hermitian) complex line bundles $L \otimes (P|Q)(\tau_{\Gamma_0}) \to (P|Q)(\tau_{\Gamma_0})$ which in turn forces the vanishing of $c[Y] = c[L]$. Thus (8.8) is free.

□

Remark 8.6 : $T(E,\omega)$ actually has rather more structure than is suggested by Proposition 8.5 - it is naturally an abelian group isomorphic to $H^2(X;\mathbb{Z})$. We shall see this shortly in Proposition 8.7 and again later in Proposition 9.13. //

Fix a $U(V)$-reduction Q of $Sp(E,\omega)$. Relative to Q , each Mp^c structure P for (E,ω) gives rise to a Hermitian line bundle $(P|Q)(\tau_{\Gamma_0})$. An equivalence $P_1 \to P_2$ of Mp^c structures induces an isomorphism $(P_1|Q)(\tau_{\Gamma_0}) \to (P_2|Q)(\tau_{\Gamma_0})$ of Hermitian line bundles. Passing to the level of equivalence classes we thus obtain

$$T:T(E,\omega) \to H^2(X;\mathbb{Z}) : [P] \mapsto c[(P|Q)(\tau_{\Gamma_0})] \quad . \tag{8.10}$$

As our notation suggests T is independent of Q .

Endow $T(E,\omega)$ with the natural $H^2(X;\mathbb{Z})$ action (8.8) and endow $H^2(X;\mathbb{Z})$ itself with the $H^2(X;\mathbb{Z})$ action coming naturally from the group structure.

Proposition 8.7 : T is a canonical isomorphism of principal $H^2(X;\mathbb{Z})$ spaces. Consequently $T(E,\omega)$ is naturally an abelian group isomorphic to $H^2(X;\mathbb{Z})$ via T .

Proof: Fix a $U(V)$ - reduction Q of $Sp(E,\omega)$. Let P be an Mp^c structure, let Y be a principal $U(1)$ bundle, and denote by L the

Hermitian line bundle associated to Y via $U(1) \to G\ell(\mathbb{C})$. The MU^c structures $(P^Y|Q)$ and $(P|Q)^Y$ are canonically equivalent : since τ_{Γ_0} is the identity on $U(1) \to MU^c(V)$, it follows that the Hermitian line bundles $(P^Y|Q)(\tau_{\Gamma_0})$ and $L \otimes (P|Q)(\tau_{\Gamma_0})$ are canonically isomorphic. Thus

$$T(c[Y].[P]) = c[Y] + T[P] \tag{8.11}$$

from which we deduce the equivariance of T. Since $\mathcal{T}(E,\omega)$ and $H^2(X;\mathbb{Z})$ are principal, T must perforce be an isomorphism. □

<u>Remark 8.8</u> : In particular, (E,ω) comes equipped with a canonical equivalence class of Mp^c structures, to which P belongs iff $(P|Q)(\tau_{\Gamma_0})$ is trivial. It is precisely this class which is considered by Plymen [Pn].

//

The twisting $(Y,P) \mapsto P^Y$ of the Mp^c structure P for (E,ω) by the principal $U(1)$ bundle Y has the following effect on the Hermitian line bundles associated to Mp^c structures via the unitary character η. We write L for the Hermitian line bundle associated to Y via $U(1) \to G\ell(\mathbb{C})$.

<u>Proposition 8.9</u> : There exists a canonical isomorphism of Hermitian line bundles

$$P^Y(\eta) \to P(\eta) \otimes L \otimes L \quad . \tag{8.12}$$

<u>Proof</u>: η restricts to $U(1) \hookrightarrow Mp^c(V,\Omega)$ as the squaring map.

□

Mp^c structures always pass down to symplectic normals. Suppose D to be an isotropic subbundle of (E,ω). Fixing a model isotropic subspace L of (V,Ω) for D, we have the $Sp(V,\Omega;L)$-reduction $Sp(E,\omega;D)$ of $Sp(E,\omega)$. If P is an Mp^c structure for (E,ω) then we denote by P^D the part of P which lies over $Sp(E,\omega;D)$; P^D is a principal $Mp^c(V,\Omega;L)$ bundle.

<u>Proposition 8.10</u> : If D is an isotropic subbundle of (E,ω) with $D \neq 0 \neq D^{\perp}/D$ and P is an Mp^c structure for (E,ω) then $(D^{\perp}/D,\omega_D)$ inherits an Mp^c structure P_D.

<u>Proof</u>: Proposition 6.7 tells us that ρ_L (1.73) lifts to Mp^c level as $\hat{\rho}_L$ (6.20). P_D is associated to P^D via $\hat{\rho}_L$. □

We close this section by considering briefly the case of Mp structures.

<u>Remark 8.11</u> : (E,ω) admits Mp structures (or, is metaplectic) iff its first Chern class is even (see Rawnsley [Ry2]). Indeed, the obstruction ($\delta[Sp(E,\omega)]$, in the sense of Proposition 8.1) to lifting $Sp(E,\omega)$ to structure group $Mp(V,\Omega)$ is precisely the second Stiefel-Whitney class $w_2(E) = \mathrm{mod}_2 c_1(E,\omega)$. //

Remark 8.12 : When nonempty, $M(E,\omega)$ is naturally a principal $H^1(X;\mathbb{Z}_2)$ space. In contrast with Proposition 8.7, however, $M(E,\omega)$ does not even have a preferred base-point in general; this may be seen as an expression of the fact that $MU^c(V)$ splits (Proposition 4.9) whereas $MU(V)$ does not (Proposition 4.10).

//

Remark 8.13 : Mp structures do not generally pass down to symplectic normals. If D is a rank r isotropic subbundle of (E,ω) then from Proposition 1.25 we deduce

$$\mathrm{mod}_2 c_1(E,\omega) = \mathrm{mod}_2 c_1(D^\perp/D,\omega_D) + w_1(D)^2 \tag{8.13}$$

since $\mathrm{mod}_2 c[\Lambda^r D^{\mathbb{C}}]$ equals the square of the first Stiefel-Whitney class $w_1(D)$. In view of Remark 8.11 it is now clear that if (E,ω) is metaplectic then $(D^\perp/D,\omega_D)$ will be metaplectic iff $w_1(D)^2 = 0$. That $w_1(D)^2$ be zero is a topological restriction. D is said to be metalinear iff $w_1(D)^2 = 0$, since this is precisely the condition that D admit an $M\ell$ structure (this being a lifting of the frame bundle of D to the metalinear group

$$M\ell(L) = \{(\lambda,g) \in \mathbb{C}^{\circ} \times G\ell(L) \mid \lambda^2 \operatorname{Det} g = 1\} \tag{8.14}$$

as structure group).

//

§9. Half-forms and Pairings.

The real symplectic vector bundle (E,ω) determines the following fibre bundles over X : $N(E,\omega)$, with fibre $N(E_x,\omega_x)$ over $x \in X$, and $n(E,\omega)$, with fibre $n(E_x,\omega_x)$ over $x \in X$. These Heisenberg bundles are of course canonically associated to the symplectic frame bundle $Sp(E,\omega)$ via the natural actions of $Sp(V,\Omega)$ on $N(V,\Omega)$ and $n(V,\Omega)$.

Let P be an Mp^c structure for (E,ω) . Associated to P via the metaplectic representation of $Mp^c(V,\Omega)$ on the rigged Hilbert space $\mathcal{E} \subset \mathbb{H} \subset \mathcal{E}'$ we have vector bundles

$$\mathcal{E}(P) \subset \mathbb{H}(P) \subset \mathcal{E}'(P) \tag{9.1}$$

of infinite rank over X . We may refer to $\mathcal{E}'(P)$ (or any of its subbundles) as a bundle of symplectic spinors for (E,ω) .

By association, the representations W (2.6), of $N(V,\Omega)$ on $\mathbb{H}$, and $\dot{W}^{\mathbb{C}}$ (2.17), of $n(V,\Omega)^{\mathbb{C}}$ on $\mathcal{E}'$, give rise to bundle representations

$$W : N(E,\omega) \to \mathrm{Aut}\,\mathbb{H}(P) \tag{9.2}$$

$$\dot{W}^{\mathbb{C}} : n(E,\omega)^{\mathbb{C}} \to \mathrm{End}\,\mathcal{E}'(P) \ . \tag{9.3}$$

The representation theory developed in earlier sections has bundle-theoretic implications which we now discuss.

Let F be a positive polarization of (E,ω). For $x \in X$ we define $E'(P)^F_x$ to be the vacuum state $(E'(P)_x)^{F_x}$ for F_x in the representation

$$\dot{W}^{\mathbb{C}}_x : n(E,\omega)^{\mathbb{C}}_x \to \mathrm{End}\, E'(P)_x \quad . \tag{9.4}$$

It is apparent from Proposition 5.4 that $E'(P)^F$ is a complex line bundle over X. $E'(P)^F$ is related to the Hermitian line bundle $P(n)$ and the canonical bundle K^F as follows:

<u>Proposition 9.1</u> : There exist canonical isomorphisms of complex line bundles

$$E'(P)^F \otimes E'(P)^F \otimes K^F \to P(n) \tag{9.5}$$

$$(E'(P)^F \otimes K^F)^2 \to P(n) \otimes K^F \quad . \tag{9.6}$$

<u>Proof</u>: $E'(P)^F$ and K^F sit (respectively) inside $E'(P)$ and $\Lambda^m(E^{\mathbb{C}})^*$, both of which are associated to P. Consequently, omitting dependence on the X-variable, we can map

$$p(f_{p^{-1}F}) \otimes p(f_{p^{-1}F}) \otimes p(k_{p^{-1}F}) \mapsto p(1) \quad ; \tag{9.7}$$

that this depends only on the $Mp^c(V,\Omega)$-orbit of $p \in P$ is clear from

Propositions 1.20 and 5.6. This defines the isomorphism (9.5) from which (9.6) follows directly.

□

Remark 9.2 : If F is regular and $\Gamma \in Lag_+(V,\Omega)$ has the type of F, then the part P^F of P which lies over $Sp(E,\omega;F) \subset Sp(E,\omega)$ is a principal $Mp^c(V,\Omega;\Gamma)$ bundle to which $E'(P)^F$ is associated via τ_Γ. In this case, an isomorphism (9.5) arises from (5.11) of Proposition 5.7.

//

Our next result tells us how the symplectic spinors (in particular, those annihilated by a positive polarization F) associated to an Mp^c structure transform under the twisting $(Y,P) \mapsto P^Y$ of an Mp^c structure P by a principal $U(1)$ bundle Y (to which the Hermitian line bundle L is associated via $U(1) \to G\ell(\mathbb{C})$) :

Proposition 9.3 : **We have a canonical isomorphism**

$$E'(P^Y) \to E'(P) \otimes L \quad . \tag{9.8}$$

In particular we have a canonical line bundle isomorphism

$$E'(P^Y)^F \to E'(P)^F \otimes L \tag{9.9}$$

for each positive polarization F .

Proof: The existence of (9.8) is clear since $U(1) \subset Mp^c(V,\Omega)$ acts trivially in the metaplectic representation. Explicitly (and omitting dependence on the X-variable) we map

$$p^y(f) \mapsto p(f) \otimes y(1) \tag{9.10}$$

for $f \in E'$ and where p^y is the natural image of $y \dot{\times} p \in Y \dot{\times} P$ in P^Y. That (9.8) restrict to (9.9) is obvious.

□

Consider for a moment a metaplectic structure P_0 for (E,ω) and denote by P the Mp^c structure associated to P_0 via inclusion $Mp(V,\Omega) \hookrightarrow Mp^c(V,\Omega)$. Since $Mp(V,\Omega)$ is the kernel of η, it is clear that $P(\eta)$ is canonically trivial. Again suppose that F is a positive polarization of (E,ω) and that Y is a principal $U(1)$ bundle determining the Hermitian line bundle L via $U(1) \to G\ell(\mathbb{C})$.

Proposition 9.4 : There exists a canonical isomorphism of complex line bundles

$$(E'(P^Y)^F \otimes K^F)^2 \to L \otimes L \otimes K^F \tag{9.11}$$

when P is associated to the Mp structure P_0 via $Mp(V,\Omega) \to Mp^c(V,\Omega)$.

Proof: (9.11) follows from Propositions 9.1 and 9.3 since $P(\eta)$ is canonically trivial.

□

Remark 9.5 : In particular, $E'(P)^F \otimes K^F$ is a canonical square-root for K^F . //

If P is an Mp^c structure for (E,ω) and F is a positive polarization of (E,ω) then we refer to the complex line bundle $E'(P)^F \otimes K^F$ as the half-form bundle for F determined by P . This terminology is suggested by Remark 9.5.

Half-forms originally arose (see Kostant [Kt2] [Kt3]) in geometric quantization as a tool in the construction of Hilbert spaces on which to represent classical observables by quantum counterparts. These Hilbert spaces are constructed using canonical sesquilinear pairings of the half-form bundles, and it is to these pairings which we now turn our attention.

We deal with the pairing of half-form bundles coming from transverse pairs of positive polarizations as follows.

Proposition 9.6 : Let P be an Mp^c structure for (E,ω) .

If (F,G) is a transverse pair of positive polarizations of (E,ω) then there exist canonical nonsingular sesquilinear pairings

$$E'(P)^F \times E'(P)^G \to \underline{\mathbb{C}} \tag{9.12}$$

$$(E'(P)^F \otimes K^F) \times (E'(P)^G \otimes K^G) \to \underline{\mathbb{C}} \ . \tag{9.13}$$

Proof: We define (9.12) as follows. Let $x \in X$ and let $p \in P_x$;

then $p^{-1}F_x$ and $p^{-1}G_x$ are positive polarizations of (V,Ω) such that $p^{-1}F_x \cap \overline{p^{-1}G_x} = 0$ so that $(f_{p^{-1}F_x}, f_{p^{-1}G_x})_0$ is defined (see Proposition 5.9). If $s \in E'(P)_x^F$ and $t \in E'(P)_x^G$ then $s = p(\alpha f_{p^{-1}F_x})$ and $t = p(\beta f_{p^{-1}G_x})$ for α and β in $\mathbb{C}$, and we define

$$\langle s,t\rangle = \alpha\bar{\beta}(f_{p^{-1}F_x}, f_{p^{-1}G_x})_0 \tag{9.14}$$

Remark 5.10 ensures that (9.14) is well-defined. (9.13) comes from (1.95) and (9.12).

□

Recall that Mp^c structures always pass to symplectic normals (Proposition 8.10). In order to extend Proposition 9.6 to cover regular pairs of positive polarizations we shall refine Proposition 8.10.

Thus, let P be an Mp^c structure for (E,ω) and D an isotropic (non-Lagrangian) subbundle of (E,ω). We write $E'(P)^D$ for the vector bundle whose fibre over $x \in X$ is the subspace of $E'(P)_x$ annihilated by D_x under $\hat{W}_x$.

<u>Proposition 9.7</u> : We have a canonical isomorphism

$$E'(P)^D \to E'(P_D) \otimes \mathcal{D}^{\frac{1}{2}}(D) \tag{9.15}$$

which restricts to a canonical isomorphism of complex line bundles

$$E'(P)^F \to E'(P_D)^{F_D} \otimes \mathcal{D}^{\frac{1}{2}}(D) \tag{9.16}$$

whenever F is a positive polarization of (E,ω) such that $D^{\mathbb{C}} \subset F$.

<u>Proof</u>: Choose a model L for D . All the bundles appearing in (9.15) are associated to the part P^D of P lying over $Sp(E,\omega;D) \subset Sp(E,\omega)$. We define (9.15) by

$$p(f) \mapsto p(R_L f) \otimes p(1) \tag{9.17}$$

for $p \in P^D$ and $f \in (E')^L$. That (9.17) well-defines (9.15) is clear from Proposition 6.7. (9.16) comes from (9.15) by virtue of Remark 6.5.

□

<u>Remark 9.8</u> : We should of course write $E'_L(P_D)$ rather than $E'(P_D)$.

//

<u>Proposition 9.9</u> : Let P be an Mp^c structure for (E,ω) . If (F,G) is a regular pair of positive polarizations of (E,ω) with $F \cap \bar{G} = D^{\mathbb{C}}$ then there exist canonical nonsingular sesquilinear pairings

$$E'(P)^F \times E'(P)^G \to \mathcal{D}^1(D) \tag{9.18}$$

$$(E'(P)^F \otimes K^F) \times (E'(P)^G \otimes K^G) \to \mathcal{D}^{-1}(D) \quad . \tag{9.19}$$

Proof: (9.19) is defined by (1.102) and (9.18). Let D be non-Lagrangian. The pair (F_D, G_D) being transverse, Proposition 9.6 provides us with a pairing $E'(P_D)^{F_D} \times E'(P_D)^{G_D} \to \underline{\mathbb{C}}$. $\mathcal{D}^{\frac{1}{2}}(D)$ naturally self-pairs into $\mathcal{D}^1(D)$. Proposition 9.7 thus determines (9.18). If D is Lagrangian then $F = G = D^{\mathbb{C}}$, and (9.18) is well-defined by mapping the pair $(p(f_{p^{-1}F}) \, , \, p(f_{p^{-1}G}))$ to $p(1) \in \mathcal{D}^1(D)$ whenever $p \in P^D$, as follows from Proposition 5.7.

□

Remark 9.10 : The symplectic form ω determines the Liouville density $|\lambda| \in \mathcal{D}^1(E)$. This canonical density gives rise to an isomorphism between $\mathcal{D}^{-1}(D)$ and $\mathcal{D}^1(E/D)$. Consequently the pairing (9.19) can be made to take values in $\mathcal{D}^1(E/D)$. We refer to Blattner [Br] and Rawnsley [Ry2] for more details on densities.

//

Remark 9.11 : The self-pairing of $P(n)$ into $\underline{\mathbb{C}}$ (determined by the Hermitian structure) links with (1.102) to give a sesquilinear pairing

$$(P(n) \otimes K^F) \times (P(n) \otimes K^G) \to \mathcal{D}^{-2}(D) \quad . \tag{9.20}$$

On the basis of Propositions 1.21 and 5.9 it is easy to see that (9.20) is the square of (9.19), in the sense determined by (9.6), when we self-pair $\mathcal{D}^{-1}(D)$ into $\mathcal{D}^{-2}(D)$.

//

In Proposition 8.7 we established that $\mathcal{T}(E,\omega)$ is naturally an abelian group isomorphic to $H^2(X;\mathbb{Z})$. We now present an alternative derivation of this result using the methods of this section.

<u>Proposition 9.12</u> : The Chern class $c[E'(P)^F]$ is independent of the positive polarization F of (E,ω) and depends only on $[P] \in \mathcal{T}(E,\omega)$.

<u>Proof</u>: Fix an Mp^c structure P . Fix a type $(0,0)$ polarization G . If F is an arbitrary positive polarization then the pair (F,G) is transverse (see Proposition 1.8); the pairing (9.12) forces an isomorphism from $E'(P)^F$ to the conjugate dual of $E'(P)^G$. Now vary P and apply Proposition 9.3.

□

<u>Proposition 9.13</u> : The natural map

$$\mathcal{T}(E,\omega) \to H^2(X;\mathbb{Z}) : [P] \mapsto c[E'(P)^F] \qquad (9.21)$$

(for F any positive polarization of (E,ω)) is an isomorphism of principal $H^2(X;\mathbb{Z})$ spaces. Consequently $\mathcal{T}(E,\omega)$ is naturally an abelian group isomorphic to $H^2(X;\mathbb{Z})$.

Proof: The equivariance of (9.21) is clear from Proposition 9.3. Since both $T(E,\omega)$ and $H^2(X;\mathbb{Z})$ are principal $H^2(X;\mathbb{Z})$ spaces, we are done. □

As might be expected, the isomorphisms (8.10) and (9.21) coincide:

Proposition 9.14 : If P is an Mp^c structure for (E,ω) and F is a positive polarization of (E,ω) then

$$c[E'(P)^F] = T[P] \quad . \tag{9.22}$$

Proof: On account of Proposition 9.12 we may suppose F to be regular of type $(0,0)$. F then corresponds canonically to a $U(V)$-reduction Q of $Sp(E,\omega)$. The complex line bundles $E'(P)^F$ and $(P|Q)(\tau_{T_0})$ are canonically isomorphic. □

Remark 9.15 : In particular the Mp^c structure P belongs to the neutral class in $T(E,\omega)$ iff $E'(P)^F$ is trivial for any (equivalently, some) positive polarization F .

//

§10. Prequantized Mp^c Structures.

Let (X,ω) be a connected symplectic manifold of dimension $2m$. We say that (X,ω) is quantizable iff the cohomology class

$$[\frac{\omega}{h}] + \tfrac{1}{2}\, c_1(TX,\omega)^{\mathbb{R}} \tag{10.1}$$

is the real image of an integer cohomology class under the coefficient change

$$(\cdot)^{\mathbb{R}} : H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}) \quad . \tag{10.2}$$

A prequantized Mp^c structure for (X,ω) is a pair (P,γ) consisting of an Mp^c structure P for (TX,ω) together with a $u(1)$-valued 1-form γ on P satisfying

$$\text{(i)} \quad a \in Mp^c(V,\Omega) \Rightarrow R_a^*\gamma = \gamma \tag{10.3}$$

$$\text{(ii)} \quad z \in mp^c(V,\Omega) \Rightarrow \gamma(\tilde{z}) = \tfrac{1}{2}\eta_* z \tag{10.4}$$

$$\text{(iii)} \qquad d\gamma = \pi^* \frac{\omega}{i\hbar} \tag{10.5}$$

where R_a is right multiplication by a, $\tilde{z}$ is the vector field generated by $R_{\exp tz}$, and $\pi : P \to X$ is the bundle projection. We say that P is prequantizable and that γ is a prequantum form.

Remark 10.1 : Observe that γ is canonically an Mp^c-invariant connexion (of curvature $\pi^* \frac{\omega}{i\hbar}$) in the principal $U(1)$ bundle $P \to Sp(TX,\omega)$.

//

The prequantized Mp^c structures (P_1,γ_1) and (P_2,γ_2) for (X,ω) are equivalent iff there exists an equivalence $f:P_1 \to P_2$ of Mp^c structures for (TX,ω) such that $f^*\gamma_2 = \gamma_1$.

Our next result relates the (unfamiliar) data of prequantized Mp^c structures to the (familiar) data of Hermitian line bundles with connexion.

Proposition 10.2 : If P is an Mp^c structure for (TX,ω) then

. prequantum forms γ on P are in natural bijective correspondence with Hermitian connexions ∇^γ of curvature $\frac{2\omega}{i\hbar}$ in the Hermitian line bundle $P(\eta)$.

Proof: Let Z denote the principal $U(1)$ bundle associated to P via η , and let $f : P \to Z$ be the associating morphism; Z is naturally the unitary frame bundle of $P(\eta)$. The bijection asserted by the Proposition is effected by

$$f^*\alpha^\gamma = 2\gamma \tag{10.6}$$

where α^γ denotes the principal connexion in Z corresponding to ∇^γ .

□

The following theorem due to Weil is of importance in the present context:

Proposition 10.3 : Let χ be a closed real 2-form on the manifold M. There exists a Hermitian line bundle $L \to M$ with metric connexion ∇ having curvature $\frac{1}{i}\chi$ iff the real cohomology class $[\frac{\chi}{2\pi}]$ is integral.

Proof: See Kostant [Kt1] .

□

We need the following corollary:

Proposition 10.4 : Let χ be a closed real 2-form on the manifold M. If L_0 is a complex line bundle over M with real Chern class

$$c[L_0]^{\mathbb{R}} = [\frac{\chi}{2\pi}] \tag{10.7}$$

then L_0 admits a Hermitian metric with compatible connexion ∇_0 of curvature $\frac{1}{i}\chi$.

Proof: Fix (L,∇) as in Proposition 10.3. The complex line bundle $L^* \otimes L_0$ has vanishing real Chern class; consequently any Hermitian metric in $L^* \otimes L_0$ has a compatible connexion of zero curvature. An isomorphism $L_0 \to L \otimes L^* \otimes L_0$ allows us to transport the Hermitian metric and compatible connexion of curvature $\frac{1}{i}X$ on $L \otimes (L^* \otimes L_0)$ (induced by tensoring those on L and $L^* \otimes L_0$) to L_0 . □

We can now decide when a given Mp^c structure is prequantizable, thus:

<u>Proposition 10.5</u> : The Mp^c structure P for (TX,ω) is prequantizable iff

$$T[P]^{\mathbb{R}} = [\frac{\omega}{h}] - \tfrac{1}{2}\, c_1(TX,\omega)^{\mathbb{R}} \quad . \tag{10.8}$$

<u>Proof</u>: From Propositions 10.2 and 10.4 we see that P is prequantizable iff

$$c[P(\eta)]^{\mathbb{R}} = [\frac{2\omega}{h}] \quad . \tag{10.9}$$

From Proposition 9.1 and 9.14 we see the equivalence of (10.8) and (10.9). □

It is now possible to give a cohomological criterion for the existence of prequantized Mp^c structures.

<u>Proposition 10.6</u> : (X,ω) admits prequantized Mp^c structures iff (X,ω) is quantizable.

Proof: Note that the automatic integrality of $c_1(TX,\omega)^{\mathbb{R}}$ means that (X,ω) is quantizable iff $[\frac{\omega}{\hbar}] - \frac{1}{2} c_1(TX,\omega)^{\mathbb{R}}$ is integral. If (X,ω) admits a prequantized Mp^c structure then (X,ω) is quantizable according to Proposition 10.5. Conversely, suppose (X,ω) to be quantizable; Proposition 8.7 ensures the existence of an Mp^c structure P for (TX,ω) such that $T[P]^{\mathbb{R}} = [\frac{\omega}{\hbar}] - \frac{1}{2}c_1(TX,\omega)^{\mathbb{R}}$; Proposition 10.5 tells us that P admits a prequantum form.

□

Remark 10.7 : Our proof of this result differs from that offered by Hess (to whom is due the concept of prequantized Mp^c structure) in [Hs]. We first produce a prequantizable Mp^c structure P and then equip it with a prequantum form γ ; Hess constructs P and γ together in one operation from local data of a Čech cohomological nature. Note that in our proof the Čech cohomology is confined to Weil's theorem.

//

In §8 we found a twisting $(Y,P) \mapsto P^Y$ of Mp^c structures by principal $U(1)$ bundles to be of importance in describing the structure of $T(E,\omega)$. We now build on this construction in order to investigate the structure of the space of equivalence classes of prequantized Mp^c structures for (X,ω) .

A flat $U(1)$ bundle over X is a pair (Y,α) consisting of a principal $U(1)$ bundle Y on X together with a flat (zero curvature) connexion α in Y . The flat $U(1)$ bundles (Y_1,α_1) and (Y_2,α_2) are equivalent iff there exists an isomorphism $f:Y_1 \to Y_2$ of principal $U(1)$ bundles such that $f^*\alpha_2 = \alpha_1$. The space of equivalence classes of flat $U(1)$ bundles over X is naturally Čech cohomology $H^1(X;U(1))$ with (locally constant) coefficients in $U(1)$.

Let (P,γ) be a prequantized Mp^c structure for (X,ω) and let (Y,α) be a flat $U(1)$ bundle over X . Recall that the Mp^c structure P^Y is associated to the fibre product $Y \dot{\times} P$. The fibre sum $\alpha \dot{+} \gamma$ of α and γ as the $u(1)$-valued form defined on $Y \dot{\times} P$ by

$$(\alpha \dot{+} \gamma)(\zeta \dot{\times} \xi) = \alpha(\zeta) + \gamma(\xi) \tag{10.10}$$

for $\zeta \dot{\times} \xi \in T(Y \dot{\times} P)$. $\alpha \dot{+} \gamma$ induces a prequantum form γ^α on P^Y by association. In this way we construct a twisting of prequantized Mp^c structures by flat $U(1)$ bundles. Passage to the level of equivalence classes gives a natural action

$$([Y,\alpha],[P,\gamma]) \mapsto [P^Y,\gamma^\alpha] \tag{10.11}$$

of $H^1(X;U(1))$ on the space of equivalence classes of prequantized Mp^c structures for (X,ω) .

Proposition 10.8: The set of equivalence classes of prequantized Mp^c structures for the quantizable (X,ω) is naturally a principal $H^1(X;U(1))$ space for the action (10.11).

Proof: Let Q be a $U(V)$-reduction of $Sp(TX,\omega)$. If (P,γ) is a prequantized Mp^c structure for (X,ω) then γ induces (by restriction and then association) a connexion in $(P|Q)(\tau_{\Gamma_0})$ which is compatible with the Hermitian structure and has curvature $\frac{\omega}{i\hbar}$. The proof now proceeds along similar lines to that of Proposition 8.5; we omit the details.

□

§11. Geometric Quantization : Prequantization.

In prequantization one aims to construct a representation module for the Poisson algebra of the symplectic manifold (X,ω) .

Such a module is constructed by Kostant [Kt1] as follows.

We say that (X,ω) is integral iff the real cohomology class $[\frac{\omega}{h}]$ is integral - thus, iff $[\frac{\omega}{h}]$ lies in the image of $H^2(X;\mathbb{Z})$ in $H^2(X;\mathbb{R})$ under $(\cdot)^{\mathbb{R}}$ (10.2).

A prequantum line bundle for (X,ω) is a Hermitian line bundle L on X equipped with a metric connexion ∇ of curvature $\frac{\omega}{i\hbar}$. The prequantum line bundles (L_1,∇_1) and (L_2,∇_2) are equivalent iff there exists an isomorphism $L_1 \to L_2$ of Hermitian line bundles mapping ∇_1 to ∇_2 .

Proposition 11.1 : (X,ω) admits prequantum line bundles iff (X,ω) is integral.

Proof: See Kostant [Kt1] (or Proposition 10.3).

□

Proposition 11.2 : When nonempty, the set of equivalence classes of prequantum line bundles for (X,ω) is naturally a principal $H^1(X;U(1))$ space.

Proof: See Kostant [Kt1] ; in principle one twists prequantum line bundles by flat $U(1)$ bundles. □

Proposition 11.3 : Let (X,ω) be integral and let (L,∇) be a prequantum line bundle for (X,ω) . A Lie algebra morphism

$$\delta : C(X) \to \operatorname{End} \Gamma(X;L) \tag{11.1}$$

is defined by

$$\delta_\phi S = \nabla_{\xi_\phi} S + \frac{1}{i\hbar} \phi S \tag{11.2}$$

for $\phi \in C(X)$ and $S \in \Gamma(X;L)$.

Proof: A straightforward computation; see Kostant [Kt1] for the details. □

Remark 11.4 : If (X,ω) is quantizable then $(X,2\omega)$ is integral and $(P(\eta),\nabla^\gamma)$ is a prequantum line bundle for $(X,2\omega)$ whenever (P,γ) is a prequantized Mp^c structure for (X,ω) . //

The prequantization scheme due to Kostant therefore represents the Poisson algebra as first-order differential operators on complex line bundles.

In addition to a choice of prequantum line bundle, the polarization - independent part of the full Kostant scheme involves a choice of metaplectic structure for (TX,ω) . The Kostant scheme thus requires both that $[\frac{\omega}{h}]$ be integral and that $c_1(TX,\omega)$ be even.

The prequantization scheme which we propose replaces the data of a prequantum line bundle together with a metaplectic structure by the data of a prequantized Mp^c structure. It is a consequence of Proposition 10.6 that our scheme will apply whenever the Kostant scheme applies.

Remark 11.5 : Indeed we can construct a prequantized Mp^c structure from Kostant data. Fix a metaplectic structure P_0 for (TX,ω) and denote by P the Mp^c structure associated to P_0 via inclusion $Mp(V,\Omega) \hookrightarrow Mp^c(V,\Omega)$. Let (L,∇) be a prequantum line bundle for (X,ω) and denote by (Y,α) the unitary frame bundle of L endowed with the connexion determined by ∇ . Then a prequantum form γ^α on P^Y is associated to $\alpha \dot{+} \vartheta$ on $Y \dot{\times} P$; compare our discussion leading up to Proposition 10.8.

//

Let us now describe our prequantization scheme in some detail. Fix a prequantized Mp^c structure (P,γ) for the quantizable symplectic manifold (X,ω) .

Recall from Remark 10.1 that γ is a connexion in the principal $U(1)$ bundle $P \to Sp(TX,\omega)$. As a consequence γ gives rise to a horizontal lift of vector fields

$$\dagger : \mathfrak{X}(Sp(TX,\omega)) \to \mathfrak{X}(P) \ . \tag{11.3}$$

Composing (11.3) with the natural Lie algebra morphism (1.108) yields a lift

$$\wedge : Ham_0(X,\omega) \to \mathfrak{X}(P) \ . \tag{11.4}$$

We shall make the natural identification of sections $s \in \Gamma(X;E'(P))$ of symplectic spinors with functions $\tilde{s} : P \to E'$ satisfying

$$p \in P \ , \ a \in Mp^c(V,\Omega) \Rightarrow \tilde{s}(p.a) = a^{-1}.\tilde{s}(p) \tag{11.5}$$

(as is standard for associated bundles - see Kobayashi & Nomizu [KN]).

<u>Proposition 11.6</u> : A map

$$D : C(X) \to \mathrm{End}\ \Gamma(X;E'(P)) \tag{11.6}$$

is defined by

$$\widetilde{(D_\phi s)} = \hat{\xi}_\phi \tilde{s} \tag{11.7}$$

and satisfies

(i) $\quad D_\phi(\psi s) = \psi D_\phi s + \{\phi,\psi\}s \qquad (11.8)$

(ii) $\quad D_{\{\phi,\psi\}}s = [D_\phi,D_\psi]s + \frac{1}{i\hbar}\{\phi,\psi\}s \qquad (11.9)$

for $\phi,\psi \in C(X)$ and $s \in \Gamma(X;E'(P))$.

<u>Proof</u>: That (11.7) well-defined D follows from (11.5) and the Mp^c-invariance of γ . (i) is clear from the fact that $\hat{\xi}_\phi$ and ξ_ϕ are π-related for the bundle projection $\pi : P \to X$. (ii) holds since (1.108) is a Lie algebra morphism and γ has curvature $\pi^* \frac{\omega}{i\hbar}$ in $P \to Sp(TX,\omega)$.

□

As a corollary we deduce

<u>Proposition 11.7</u> : A Lie algebra morphism

$$\delta : C(X) \to \mathrm{End}\, \Gamma(X,E'(P)) \tag{11.10}$$

is defined by the prescription

$$\delta_\phi s = D_\phi s + \frac{1}{i\hbar} \phi s \tag{11.11}$$

for $\phi \in C(X)$ and $s \in \Gamma(X;E'(P))$.

<u>Proof</u>: A straightforward consequence of (11.8) and (11.9); we omit the details.

□

We shall refer to the Lie algebra morphism δ of Proposition 11.7 as prequantization of (X,ω) relative to (P,γ) .

<u>Remark 11.8</u> : Equivalent prequantized Mp^c structures yield equivalent prequantizations : an equivalence $f:(P_1,\gamma_1) \to (P_2,\gamma_2)$ of prequantized

Mp^c structures gives rise naturally to an isomorphism $f_*: E'(P_1) \to E'(P_2)$ which satisfies $\delta^2_\phi \circ f_* = \delta^1_\phi$ for $\phi \in C(X)$, where δ^1 , δ^2 are prequantizations relative to (P_1, γ_1) , (P_2, γ_2) . Notice that Proposition 10.8 tells us that the inequivalent prequantization data for (X, ω) are parametrized by $H^1(X; U(1))$.

//

It is both important in theory and convenient in practice to have available a local picture of prequantization. According to the well-known Darboux theorem on the universal existence of local symplectic coordinate charts, every symplectic manifold is locally linear. For a local description of prequantization (and indeed of full quantization) we therefore refer to §13, where we analyze in detail the case of linear symplectic manifolds.

The bundle representation $\dot{W}^{\mathbb{C}}$ of $n(TX, \omega)^{\mathbb{C}}$ on $E'(P)$ (see (9.3)) naturally induces a map (also denoted $\dot{W}^{\mathbb{C}}$) from $\mathfrak{X}(X) \leftrightarrow \Gamma(X; n(TX, \omega)^{\mathbb{C}})$ to End $\Gamma(X; E'(P))$. Our next result provides an important relationship between this map and the map D (11.6); its proof amply illustrates the usefulness of our local picture of prequantization.

<u>Proposition 11.9</u> : If $\phi \in C(X)$ and $\xi \in \mathfrak{X}(X)$ then

$$[D_\phi, \dot{W}^{\mathbb{C}}(\xi)] = \dot{W}^{\mathbb{C}}[\xi_\phi, \xi] \quad . \tag{11.12}$$

<u>Proof</u>: It suffices to establish the formula locally; this is done by transporting formula (13.23) of Proposition 13.8 via a local symplectic chart.

□

§12. Geometric Quantization : Polarization.

In quantizing the symplectic manifold (or, phase space) (X,ω) one aims to represent smooth functions on X (or, classical observables) as densely-defined essentially skew-adjoint linear operators (or, quantum observables) on some Hilbert space, in such a way that the Poisson bracket passes across to the commutator bracket.

The Kostant scheme achieves this by tensoring a prequantum line bundle with the bundle of half-forms for a positive polarization defined by a metaplectic structure, and then constructing a Hilbert space by pairing sections of the resulting line bundle. We refer to Kostant [Kt3] for details of the procedure.

In the preceding section we saw that our scheme replaces the two sets of data (choices of prequantum line bundle and metaplectic structure) required by the Kostant scheme with one set of data (a choice of prequantized Mp^c structure (P,γ) for (X,ω)), and we constructed a Lie algebra morphism

$$\delta : C(X) \to \operatorname{End} \Gamma(X;E'(P)) \qquad (12.1)$$

which we called prequantization of (X,ω) relative to (P,γ).

In the present section we shall see that a choice of positive polarization F of (X,ω) picks out from δ subrepresentations on spaces of sections of tne line bundle $E'(P)^F$ of vacuum states and that tensoring with the Lie derivative in the canonical bundle K^F gives a representation of $C^1_F(X)$ on polarized sections of the half-form bundle $E'(P)^F \otimes K^F$. The half-form pairings of §9 will then enable us to quantize (X,ω) relative to the quantization data $(P,\gamma;F)$ and to compare the quantizations which arise by varying F. All of this will take place under suitable regularity hypotheses.

Let us now describe our full quantization scheme in detail. Let (X,ω) be a quantizable symplectic manifold. Fix a prequantized Mp^c structure (P,γ) for (X,ω). Choose a positive polarization F of (X,ω).

Lie differentiation in the bundle $\Lambda^m T^* X^{\mathbb{C}}$ of complex m-forms composes with ξ (1.104) to yield a Lie algebra morphism

$$C(X) \to \mathrm{End}\,\Gamma(X;\Lambda^m T^* X^{\mathbb{C}}) : \phi \mapsto L_{\xi_\phi} \tag{12.2}$$

If $\phi \in C^1_F(X)$ then (since F is involutive) L_{ξ_ϕ} stabilizes $\Gamma(X;K^F) \subset \Gamma(X;\Lambda^m T^* X^{\mathbb{C}})$. Consequently (12.2) induces a Lie algebra morphism

$$C^1_F(X) \to \mathrm{End}\,\Gamma(X;K^F) : \phi \mapsto L_{\xi_\phi} \quad . \tag{12.3}$$

These remarks on the canonical bundle have the following analogue for the bundle of vacuum states.

<u>Proposition 12.1</u> : If $\phi \in C^1_F(X)$ and $s \in \Gamma(X;E'(P)^F)$ then $D_\phi s$ and $\delta_\phi s$ lie in $\Gamma(X;E'(P)^F)$.

<u>Proof</u>: Clear from Proposition 11.9.

□

Thus prequantization restricts to give a Lie algebra morphism

$$\delta : C^1_F(X) \to \mathrm{End}\ \Gamma(X;E'(P)^F) \tag{12.4}$$

which upon tensoring with Lie differentiation in K^F yields a Lie algebra morphism

$$\delta^F : C^1_F(X) \to \mathrm{End}\ \Gamma(X;E'(P)^F \otimes K^F) \tag{12.5}$$

given by the prescription

$$\phi \in C^1_F(X) \Rightarrow \delta^F_\phi = \delta_\phi \otimes I + I \otimes L_{\xi_\phi} . \tag{12.6}$$

In similar fashion D gives rise to

$$D^F : C^1_F(X) \to \mathrm{End}\ \Gamma(X;E'(P)^F \otimes K^F) \tag{12.7}$$

defined by

$$\phi \in C^1_F(X) \Rightarrow D^F_\phi = D_\phi \otimes I + I \otimes L_{\xi_\phi} \tag{12.8}$$

and satisfying the $(\cdot)^F$-analogues of (11.8) (11.9). δ^F and D^F

are clearly related by

$$\delta^F_\phi s = D^F_\phi s + \frac{1}{i\hbar}\,\phi s \tag{12.9}$$

for $\phi \in C^1_F(X)$ and $s \in \Gamma(X;E'(P)^F \otimes K^F)$.

The space of all sections of the half-form bundle is too large for the purposes of quantization; we cut it down to the space of polarized sections as follows.

The section s of $E'(P)^F \otimes K^F$ is said to be polarized iff

$$X \in C_F(U) \Rightarrow D^F_X s = 0 \tag{12.10}$$

whenever U is an open set in X . We denote by $\Gamma_0(E'(P)^F \otimes K^F)$ (or by $\Gamma_0(X;E'(P)^F \otimes K^F)$) the space of all polarized sections of $E'(P)^F \otimes K^F$.

Let $U \subset X$ be open. $\Gamma_0(U;E'(P)^F \otimes K^F)$ is clearly a $\mathbb{C}$-linear subspace of $\Gamma(U;E'(P)^F \otimes K^F)$. More is true: $\Gamma_0(U;E'(P)^F \otimes K^F)$ is naturally a $C_F(U)$-module for the standard action of functions on sections.

<u>Proposition 12.2</u> : Let $U \subset X$ be open. If $\phi \in C_F(U)$ and $s \in \Gamma_0(U;E'(P)^F \otimes K^F)$ then $\phi s \in \Gamma_0(U;E'(P)^F \otimes K^F)$.

<u>Proof</u>: If $\psi \in C_F(U)$ then

$$D^F_\psi(\phi s) = \phi D^F_\psi s + \{\psi,\phi\}s$$

vanishes since $C_F(U)$ is abelian.

□

Still more is true: $\Gamma_0(E'(P)^F \otimes K^F)$ is stable under the action of C^1_F determined by δ^F .

<u>Proposition 12.3</u> : If $\phi \in C^1_F(X)$ and $s \in \Gamma_0(X;E'(P)^F \otimes K^F)$ then $\delta^F_\phi s \in \Gamma_0(X;E'(P)^F \otimes K^F)$.

<u>Proof</u>: If $U \subset X$ is open and $\psi \in C_F(U)$ then

$$\begin{aligned} D^F_\psi(\delta^F_\phi s) &= D^F_\psi(D^F_\phi s) + D^F_\psi(\tfrac{1}{i\hbar}\phi s) \\ &= D^F_\phi D^F_\psi s + D^F_{\{\psi,\phi\}}s - \tfrac{1}{i\hbar}\{\psi,\phi\}s + \tfrac{1}{i\hbar}\phi D^F_\psi s + \tfrac{1}{i\hbar}\{\psi,\phi\}s \end{aligned}$$

which vanishes since $C_F(U)$ is an ideal in $C^1_F(U)$.

□

By restriction we therefore have a Lie algebra morphism

$$\delta^F : C^1_F(X) \to \mathrm{End}\,\Gamma_0(X;E'(P)^F \otimes K^F) \tag{12.11}$$

which may be called quantization of (X,ω) relative to the quantization data $(P,\gamma;F)$. We may refer to the half-form bundle $E'(P)^F \otimes K^F$ as the quantum bundle for the data $(P,\gamma;F)$.

Remark 12.4 : Note that (by virtue of (1.96) and Propositions 9.1, 9.14, and 10.5) the real Chern class of $E'(P)^F \otimes K^F$ is given by

$$c[E'(P)^F \otimes K^F]^{\mathbb{R}} = [\tfrac{\omega}{h}] + \tfrac{1}{2}c_1(TX,\omega)^{\mathbb{R}} \tag{12.12}$$

as has come to be expected of a quantum bundle.

//

Recall from Proposition 9.1 that we have a canonical isomorphism

$$(E'(P)^F \otimes K^F)^2 \to P(n) \otimes K^F \tag{12.13}$$

of complex line bundles. In $E'(P)^F \otimes K^F$ we have the operator D^F, in $P(n)$ the metric connexion ∇^γ, and in K^F the Lie derivative. These are related by the following Leibnitz rule relative to (12.13).

Proposition 12.5 : If $\phi \in C^1_F(X)$ then

$$D^F_\phi \otimes I + I \otimes D^F_\phi = \nabla^\gamma_{\xi_\phi} \otimes I + I \otimes L_{\xi_\phi} \quad . \tag{12.14}$$

Proof: It suffices to give a local verification, for which see Proposition 13.12. □

Remark 12.6 : This result has the following important practical consequence. If we are able to identify both $(P(n),\nabla^\gamma)$ and the square-root $E'(P)^F \otimes K^F$ of $P(n) \otimes K^F$, then in order to compute the quantization we simply pass $\nabla^\gamma \otimes I + I \otimes L$ to the square-root (uniquely) via the Leibnitz rule (and add the appropriate multiplication operator).

//

As we have presented it thus far, our geometric quantization scheme is perhaps rather abstract. Let us cast the scheme into a more familiar setting: that of complex line bundles equipped with flat partial connexions. We recall below the definition of a partial connexion and its flatness; for further information see Rawnsley [Ryl].

If A is a complex vector bundle over the manifold M and B is a complex vector subbundle of $TM^{\mathbb{C}}$ then a B-connexion in A (or, a partial connexion in A defined along B) is a map

$$\Gamma(M;B) \times \Gamma(M;A) \to \Gamma(M;A):(b,a) \mapsto \nabla_b a \tag{12.15}$$

which satisfies $\nabla_{\phi b} a = \phi \nabla_b a$ and the Leibnitz rule

$$\nabla_b(\phi a) = \phi \nabla_b a + (b\phi)a \tag{12.16}$$

for $\phi \in C(M), a \in \Gamma(M;A), b \in \Gamma(M;B)$ (and where b acts on ϕ as a vector field).

If $B \subset TM^{\mathbb{C}}$ is involutive in the sense

$$b_1, b_2 \in \Gamma(M;B) \subset \mathfrak{X}(M) \Rightarrow [b_1, b_2] \in \Gamma(M;B) \tag{12.17}$$

then we say that ∇ is flat iff

$$b_1, b_2 \in \Gamma(M;B) \Rightarrow \nabla_{[b_1, b_2]} = [\nabla_{b_1}, \nabla_{b_2}] \ . \tag{12.18}$$

Thus ∇ is flat iff as much of its curvature as can be defined is zero.

<u>Remark 12.7</u> : Let us suppose the positive polarization F to be strongly regular; thus X is covered by local C_F-charts (see §1). We claim that the operator D^F (12.7) gives rise to a flat F-connexion ∇^F in $E'(P)^F \otimes K^F$. To see this, let $\zeta \in \Gamma(X;F)$. Let $(\psi_1, \ldots, \psi_m)$ be a C_F-chart over the open set $U \subset X$, so that $\{\xi_{\psi_1}, \ldots, \xi_{\psi_m}\}$ spans $\Gamma(U;F)$ over $C(U)$. Then there exist $\phi_1, \ldots, \phi_m$ in $C(U)$ such that

$$\zeta = \sum_{j=1}^{m} \phi_j \xi_{\psi_j} \ . \tag{12.19}$$

For $s \in \Gamma(X; E'(P)^F \otimes K^F)$ we define

$$\nabla_\zeta^F s = \sum_{j=1}^{m} \phi_j D_{\psi_j}^F s \ . \tag{12.20}$$

In view of Proposition 11.6 (i) and the fact that X is covered by local C_F-charts, it is clear that this procedure well-defines an F-connexion ∇^F in $E'(P)^F \otimes K^F$. Since $C_F(X)$ is abelian, it is clear from Proposition 11.6 (ii) that ∇^F is flat. Note that the polarized sections of $E'(P)^F \otimes K^F$ are determined by ∇^F after the usual fashion for flat partial connexions: $s \in \Gamma(X;E'(P)^F \otimes K^F)$ is polarized iff

$$\zeta \in \Gamma(X;F) \Rightarrow \nabla^F_\zeta s = 0 \; . \qquad (12.21).$$

From Proposition 12.5 we deduce that

$$\zeta \in \Gamma(X;F) \Rightarrow \nabla^F_\zeta \otimes I + I \otimes \nabla^F_\zeta = \nabla^\gamma_\zeta \otimes I + I \otimes L_\zeta \qquad (12.22)$$

relative to the isomorphism (12.13).

//

Remark 12.8 : As a special case, suppose (X,ω) to be the symplectic manifold which underlies a Kähler manifold and suppose F to be the bundle of antiholomorphic tangents. C_F is then the sheaf of (germs of) holomorphic functions and K^F the bundle of holomorphic m-forms. If P is a prequantizable Mp^c structure then each prequantum form γ on P endows $E'(P)^F \otimes K^F$ with a flat F-connexion ∇^F (as in Remark 12.7) and according to Rawnsley [Ry3] there is a unique holomorphic structure in $E'(P)^F \otimes K^F$ which is compatible with ∇^F in the sense that the (local) holomorphic sections are precisely the (local ∇^F-) polarized sections. $P(\eta)$ is likewise given a holomorphic structure and (12.13) is a holomorphic isomorphism when K^F has the canonical holomorphic structure.

//

In accordance with the aim expressed at the start of this section we should indicate how geometric quantization develops from this point onwards; in fact the subsequent development differs little from that of the established scheme due to Blattner, Kostant, Sternberg, ... We outline the procedure.

For each involutive isotropic subbundle D of TX appearing below we make the following assumptions:

(a) The foliation D is fibrating; thus, the leaf-space X/D is a manifold and the projection $X \to X/D$ a submersion.

(b) Blattner's obstruction (to passing pairings down to X/D) vanishes; see Blattner [Br] and Rawnsley [Ry2].

We remark that (b) will hold whenever $D^\perp$ is involutive (which will be the case if D arises from a strongly regular pair (F,G) of positive polarizations of (X,ω) as $F \cap \bar{G} = D^{\mathbb{C}}$) . These assumptions are needed to facilitate the passage to Hilbert spaces.

Let F be a regular positive polarization of (X,ω) with $F \cap \bar{F} = D_F^{\mathbb{C}}$. Referring to Remark 9.10 we have a self-pairing $\langle\langle\cdot,\cdot\rangle\rangle_F$ of the quantum bundle $E'(P)^F \otimes K^F$ into $\mathcal{D}^1(TX/D_F)$. If s and t are polarized sections of $E'(P)^F \otimes K^F$ then $\langle\langle s,t\rangle\rangle_F$ descends to a density $\langle s,t\rangle_F \in \mathcal{D}^1(X/D_F)$ on the leaf-space X/D_F . Denote by H_F the subspace of $\Gamma_0(E'(P)^F \otimes K^F)$ consisting of those s for which $\langle s,s\rangle_F \in \mathcal{D}^1(X/D_F)$ has compact support (and so may be integrated over X/D_F) . H_F is a pre-Hilbert space with inner product $\int\langle\cdot,\cdot\rangle_F$ and is stable under δ^F_ϕ for $\phi \in C^1_F(X)$ since δ^F_ϕ is support-decreasing. The completion $\mathbb{H}_F = \mathbb{H}_F(X,\omega;P,\gamma)$ of H_F is the Hilbert space on which we quantize $C^1_F(X)$.

It is desirable to be able to compare the quantizations arising from a pair of positive polarizations; our scheme caters for this.

Let (F,G) be a regular pair of regular positive polarizations of (X,ω) . Let $F \cap \bar{F} = D^{\mathbb{C}}_F$, $G \cap \bar{G} = D^{\mathbb{C}}_G$, and $F \cap \bar{G} = D^{\mathbb{C}}_{F,G}$; Proposition 1.8 implies that $D_{F,G} = D_F \cap D_G$. From Remark 9.10 we have a pairing $\langle\langle\cdot,\cdot\rangle\rangle_{F,G}$ of $E'(P)^F \otimes K^F$ and $E'(P)^G \otimes K^G$ into $\mathcal{D}^1(TX/D_{F,G})$. Under favourable conditions, $\langle\langle\cdot,\cdot\rangle\rangle_{F,G}$ gives rise to a pairing $\int\langle\cdot,\cdot\rangle_{F,G}$ of $\mathbb{H}_F$ and $\mathbb{H}_G$.

§13. Linear Symplectic Manifolds.

In this section we show how the theory of geometric quantization developed in the preceding sections works out in practice in the case of a linear symplectic manifold. In addition to serving as an illustration of some of our techniques, the case of a linear symplectic manifold provides a convenient local picture of the scheme.

Let (V,Ω) be a real $2m$-dimensional symplectic vector space. X will denote V endowed with the natural manifold structure. For each $x \in X$ there is a natural linear isomorphism

$$b_x : V \to T_xX : v \mapsto v_x \tag{13.1}$$

given by

$$v_x f = \frac{d}{dt} f(x + tv)\Big|_{t=0} \tag{13.2}$$

for $f \in C(X)$. A symplectic form ω on X is defined by transport of Ω via b - thus

$$\omega_x(b_xv_1,b_xv_2) = \Omega(v_1,v_2) \tag{13.3}$$

for $x \in X$ and $v_1,v_2 \in V$. We refer to (X,ω) as the linear symplectic manifold modelled on (V,Ω).

Observe that $Sp(TX,\omega)$ is canonically trivial when modelled on (V,Ω) - indeed a (global) trivialization is defined by

$$B:X \times Sp(V,\Omega) \to Sp(TX,\omega):(x,g) \mapsto b_x \circ g \quad . \tag{13.4}$$

Since $H^2(X;\mathbb{Z}) = 0$ it follows from Proposition 8.7 that there is (up to equivalence) precisely one Mp^c structure for (TX,ω) . We shall fix as Mp^c structure the product principal $Mp^c(V,\Omega)$ bundle $P = X \times Mp^c(V,\Omega)$ with the composite

$$X \times Mp^c(V,\Omega) \xrightarrow{I\times\sigma} X \times Sp(V,\Omega) \xrightarrow{B} Sp(TX,\omega) \tag{13.5}$$

as covering morphism.

(X,ω) is quantizable; indeed, since $H^2(X;\mathbb{R}) = 0$, Proposition 10.5 guarantees that all Mp^c structures are prequantizable. Moreover, since $H^1(X;U(1))$ is trivial, it follows from Proposition 10.8 that all prequantized Mp^c structures for (X,ω) are equivalent.

If we let α_0 denote the natural flat connexion in the product principal $Mp^c(V,\Omega)$ bundle $\pi:P \to X$, then we have the following description of prequantum forms supported by P .

<u>Proposition 13.1</u> : The $u(1)$-valued 1-form γ on P is a prequantum form for (X,ω) iff

$$\gamma = \tfrac{1}{2}\eta_*\alpha_0 + \frac{1}{i\hbar}\pi^*\theta \tag{13.6}$$

for some primitive θ of ω .

Proof: An elementary exercise in basic forms, which we omit.

□

Having obtained explicit forms for the prequantization data we now turn to the determination of an explicit form for prequantization

$$\delta : C(X) \to \mathrm{End}\,\Gamma(X;E'(P)) \tag{13.7}$$

relative to $(P, \gamma = \frac{1}{2}\eta_*\alpha_0 + \frac{1}{i\hbar}\pi^*\theta)$.

Let α^b be the flat connexion in $Sp(TX,\omega) \to X$ determined by the natural flat connexion in the product $X \times Sp(V,\Omega) \to X$ via B ; we use the symbol α^b also to denote its complexification.

If $\phi \in C(X)$ then

$$z_\phi = \alpha^b(\tilde{\xi}_\phi) \circ b \tag{13.8}$$

is an $sp(V,\Omega)^{\mathbb{C}}$-valued function on X ; this gives us a map

$$z : C(X) \to C(X) \otimes sp(V,\Omega)^{\mathbb{C}} . \tag{13.9}$$

Let ϕ be a real-valued function on X . If ξ_ϕ generates the local 1-parameter group σ^t of symplectic automorphisms of (X,ω) , then the local 1-parameter group $\tilde{\sigma}^t$ of automorphisms of $Sp(TX,\omega)$ generated by $\tilde{\xi}_\phi$ is given by

$$\tilde{\sigma}^t(c) = \sigma_*^t \circ c \tag{13.10}$$

for $c \in Sp(TX,\omega)$ and satisfies

$$\tilde{\sigma}^t(b_x) = \sigma_*^t \circ b_x = b_{\sigma^t x} \circ \exp tz_\phi(x) \tag{13.11}$$

for $x \in X$.

Identifying the tangent space to a product with the product of the tangent spaces, we have:

<u>Proposition 13.2</u> : If $\phi \in C(X)$ and $(x,g) \in X \times Sp(V,\Omega)$ then

$$(\tilde{\xi}_\phi)_{B(x,g)} = B_*((\xi_\phi)_x,(\text{Ad } g^{-1}z_\phi(x))_g) \quad . \tag{13.12}$$

<u>Proof</u>: If ϕ is $\mathbb{R}$-valued then (13.12) is a routine consequence of (13.11); the general case follows by complexification. □

Relative to the canonical splitting (7.3) of $mp^c(V,\Omega)$, we now have the following description of $\wedge\circ\xi$ (11.4) (1.104):

<u>Proposition 13.3</u> : If $\phi \in C(X)$ and $(x,a) \in X \times Mp^c(V,\Omega)$ with $\sigma(a) = g$ then

$$\hat{\xi}_\phi(x,a) = ((\xi_\phi)_x , ((-\tfrac{1}{i\hbar}(\theta\xi_\phi)_x) \oplus \text{Ad } g^{-1}z_\phi(x))_a) \quad . \tag{13.13}$$

<u>Proof</u>: Since $\hat{\xi}_\phi$ lifts $\tilde{\xi}_\phi$ it is clear that

$$\hat{\xi}_\phi(x,a) = ((\xi_\phi)_x , (\zeta \oplus \text{Ad } g^{-1}z_\phi(x))_a)$$

for some $\zeta \in u(1)^{\mathbb{C}}$, from Proposition 13.2. The γ-horizontality of $\hat{\xi}_\phi$ enables us to identify

$$\zeta = -\frac{1}{i\hbar}(\theta\xi_\phi)_x \ .$$

$\square$

Since P is the product $X \times Mp^c(V,\Omega)$ we have a canonical trivialization

$$E'(P) \to X \times E' : [(x,a),f] \mapsto (x,\mu(a)f) \tag{13.14}$$

and hence a canonical identification

$$s:C(X) \otimes E' \to \Gamma(X;E'(P)) : f \mapsto s_f \tag{13.15}$$

of E'-valued functions with sections of $E'(P)$.

Regarding $sp(V,\Omega)$ as $mp(V,\Omega)$ via (7.3) we have

<u>Proposition 13.4</u> : If $\phi \in C(X)$ and $f \in C(X) \otimes E'$ then

$$D_\phi(s(f)) = s(\frac{1}{i\hbar}(\theta\xi_\phi)f - \dot{\mu}^{\mathbb{C}}(z_\phi)f + \xi_\phi f) \tag{13.16}$$

<u>Proof</u>: A routine consequence of Proposition 13.3.

$\square$

As a corollary we give explicit form to prequantization δ of (X,ω) relative to $(P,\gamma = \frac{1}{2}n_*\alpha_0 + \frac{1}{i\hbar}\pi^*\theta)$.

Proposition 13.5 : If $\phi \in C(X)$ and $f \in C(X) \otimes E'$ then

$$\delta_\phi(s(f)) = s(\tfrac{1}{i\hbar}(\phi+\theta\xi_\phi)f - \dot{\mu}^{\mathbb{C}}(z_\phi)f + \xi_\phi f) \ . \tag{13.17}$$

Proof: Immediate from Proposition 13.4. □

Remark 13.6 : This formula for prequantization compares with that given by Kostant in [Kt3], modulo notational conventions and the term in $\frac{1}{i\hbar}(\theta\xi_\phi)$ (which comes naturally from the structure of a prequantum line bundle on $\underline{\mathbb{C}}$). //

Since $TX^{\mathbb{C}}$ is canonically isomorphic to $X \times V^{\mathbb{C}}$ we have a canonical identification

$$\zeta : C(X) \otimes V^{\mathbb{C}} \to \mathfrak{X}(X) : v \mapsto \zeta_v \tag{13.18}$$

of $V^{\mathbb{C}}$-valued functions with complex vector fields.

Proposition 13.7 : Let $\phi \in C(X)$, $v \in C(X) \otimes V^{\mathbb{C}}$, and $f \in C(X) \otimes E'$; then

(i) $\dot{W}^{\mathbb{C}}(\zeta_v)(s_f) = s(\dot{W}^{\mathbb{C}}(v)f)$ (13.19)

(ii) $\zeta(\xi_\phi v - z_\phi \cdot v) = [\xi_\phi, \zeta_v]$ (13.20)

and, as operators on $C(X) \otimes E'$,

(iii) $[\xi_\phi, \dot{W}^{\mathbb{C}}(v)] = \dot{W}^{\mathbb{C}}(\xi_\phi v)$ (13.21)

(iv) $[\dot{\mu}^{\mathbb{C}}(z_\phi), \dot{W}^{\mathbb{C}}(v)] = \dot{W}^{\mathbb{C}}(z_\phi \cdot v)$. (13.22)

Proof: (i) is clear. If ϕ is real then (ii) comes by differentiation along the flow generated by ξ_ϕ (making use of (13.11)); the general case follows by complexification. Similar computations yield (iii). Finally, (iv) is a consequence of Proposition 7.8.

□

We are now able to establish:

Proposition 13.8 : If $\phi \in C(X)$ and $\xi \in \mathfrak{X}(X)$ then

$$[D_\phi, \dot{W}^{\mathbb{C}}(\xi)] = \dot{W}^{\mathbb{C}}[\xi_\phi, \xi] \quad . \tag{13.23}$$

Proof: If $v \in C(X) \otimes V^{\mathbb{C}}$ and $f \in C(X) \otimes E'$ then Proposition 13.7 justifies the following:

$$\begin{aligned} D_\phi \dot{W}^{\mathbb{C}}(\zeta_v) s_f &= D_\phi s(\dot{W}^{\mathbb{C}}(v) f) \\ &= s\left(\left(\tfrac{1}{i\hbar}(\theta \xi_\phi) - \dot{\mu}^{\mathbb{C}}(z_\phi) + \xi_\phi\right)(\dot{W}^{\mathbb{C}}(v) f)\right) \\ &= s\left(\dot{W}^{\mathbb{C}}(v)\left(\tfrac{1}{i\hbar}(\theta \xi_\phi) - \dot{\mu}^{\mathbb{C}}(z_\phi) + \xi_\phi\right) f\right) \\ &\quad + s(-\dot{W}^{\mathbb{C}}(z_\phi \cdot v) f + \dot{W}^{\mathbb{C}}(\xi_\phi v) f) \\ &= \dot{W}^{\mathbb{C}}(\zeta_v) D_\phi s_f + \dot{W}^{\mathbb{C}}[\xi_\phi, \zeta_v] s_f \quad . \end{aligned}$$

□

As we pointed out in Remark 11.4, (P,γ) gives rise to a prequantum line bundle $(P(\eta),\nabla^{\gamma})$ for $(X,2\omega)$. Let us determine ∇^{γ} explicitly by computing its effect on the (canonical unitary) section $t \in \Gamma(X;P(\eta))$ given by

$$t : X \to P(\eta) : x \mapsto [(x,I),1] \quad . \tag{13.24}$$

<u>Proposition 13.9</u> : If $\phi \in C(X)$ then

$$\nabla^{\gamma}_{\xi_\phi} t = \frac{2}{i\hbar}(\theta\xi_\phi)t \quad . \tag{13.25}$$

<u>Proof</u>: An application of Proposition 13.3.

□

We now turn to quantization proper. Again we begin by considering Lie differentiation in canonical bundles.

Since $\Lambda^m T^* X^{\mathbb{C}}$ is canonically isomorphic to $X \times \Lambda^m(V^{\mathbb{C}})^*$ we have a canonical identification

$$s : C(X) \otimes \Lambda^m(V^{\mathbb{C}})^* \to \Gamma(X;\Lambda^m T^* X^{\mathbb{C}}) : k \mapsto s_k \tag{13.26}$$

of $\Lambda^m(V^{\mathbb{C}})^*$-valued functions with complex m-forms. In terms of the natural representation of $sp(V,\Omega)^{\mathbb{C}}$ on $\Lambda^m(V^{\mathbb{C}})^*$ we have:

<u>Proposition 13.10</u> : If $\phi \in C(X)$ and $k \in C(X) \otimes \Lambda^m(V^{\mathbb{C}})^*$ then

$$L_{\xi_\phi} s_k = s(\xi_\phi k - z_\phi \cdot k) \quad . \tag{13.27}$$

Proof: A straightforward consequence of Proposition 13.2 since $\Lambda^m T^* X^{\mathbb{C}}$ is canonically associated to $Sp(TX,\omega)$.

□

Remark 13.11 : Since the canonical bundle of a polarization of (X,ω) is a complex line bundle in $\Lambda^m T^* X^{\mathbb{C}}$, Proposition 13.10 tells us (at least in principle) how to Lie differentiate in canonical bundles.

//

Let F be a positive polarization of (X,ω) . For each $x \in X$ we let $\Gamma_x \in Lag_+(V,\Omega)$ be given by

$$b_x^{\mathbb{C}} \Gamma_x = F_x \quad . \tag{13.28}$$

Defining $f \in C(X) \otimes E'$ by

$$f : X \to E' : x \mapsto f_{\Gamma_x} \tag{13.29}$$

we have a canonical (zerofree) section s_f of $E'(P)^F$ and defining $k \in C(X) \otimes \Lambda^m(V^{\mathbb{C}})^*$ by

$$k : X \to \overset{\Lambda^m(V^{\mathbb{C}})^*}{\cancel{E'}} : x \mapsto k_{\Gamma_x} \tag{13.30}$$

we have a canonical (zero-free) section s_k of K^F ; $s_{f \otimes k} = s_f \otimes s_k$ is then a canonical (zero-free) section of the half-form bundle $E'(P)^F \otimes K^F$. In terms of these special sections $t \in \Gamma(X;P(n))$, s_f and s_k , the canonical isomorphism

$$(E'(P)^F \otimes K^F)^2 \to P(n) \otimes K^F \tag{13.31}$$

of complex line bundles (provided by Proposition 9.1) is determined by

$$(s_{f\otimes k})^2 \mapsto t\otimes s_k \quad . \tag{13.32}$$

We have the following Leibnitz rule relative to (13.31).

<u>Proposition 13.12</u> : If $\phi \in C^1_F(X)$ then

$$D^F_\phi \otimes I + I \otimes D^F_\phi = \nabla^\gamma_{\xi_\phi} \otimes I + I \otimes L_{\xi_\phi} \quad . \tag{13.33}$$

<u>Proof</u>: By explicit evaluation, of $D^F_\phi \otimes I + I \otimes D^F_\phi$ on $(s_f \otimes s_k)^2$ and of $\nabla^\gamma_{\xi_\phi} \otimes I + I \otimes L_{\xi_\phi}$ on $t \otimes s_k$; we give a sketch. Let $\phi = r + is$ with r and s real-valued. For $x \in X$ we write

$$(\lambda_t(x), g_t(x)) = \exp tz_r(x) \in Mp(V,\Omega)$$

$$(\mu_t(x), h_t(x)) = \exp tz_s(x) \in Mp(V,\Omega) \quad .$$

Proposition 5.6 tells us that if $x \in X$ then

$$\begin{aligned} \dot{\mu}^{\mathbb{C}}(z_\phi(x))f_{\Gamma_x} &= \frac{d}{dt}\{\lambda_t(x)\mathrm{Det}^{\frac{1}{2}}(I+Z_{g_t(x)}Z_{\Gamma_x})^{-1}f_{g_t(x).\Gamma_x} \\ &\quad + i\mu_t(x)\mathrm{Det}^{\frac{1}{2}}(I+Z_{h_t(x)}Z_{\Gamma_x})^{-1}f_{h_t(x).\Gamma_x}\}\Big|_{t=0} \\ &= -\tfrac{1}{2}y_\phi(x)f_{\Gamma_x} + \frac{d}{dt}\{f_{g_t(x).\Gamma_x} + if_{h_t(x).\Gamma_x}\}\Big|_{t=0} \end{aligned}$$

where (with the aid of Proposition 7.5)

$$y_\phi(x) = \mathrm{Tr}(C_{z_r}(x) + A_{z_r}(x)Z_{\Gamma_x}) + i\mathrm{Tr}(C_{z_s}(x) + A_{z_s}(x)Z_{\Gamma_x}) \tag{13.34}$$

According to Propositions 13.4 and 13.8 we now have

$$D_\phi s_f = (\tfrac{1}{i\hbar}(\theta\xi_\phi) + \tfrac{1}{2}y_\phi)s_f \tag{13.35}$$

since differentiating the constraint

$$x \in X \Rightarrow (f_{\Gamma_x}, f_{\Gamma_0})_0 = 1$$

of Proposition 5.9 tells us (as in Remark 7.12)

$$x \in X \Rightarrow (\xi_\phi f)_x - \frac{d}{dt}\{f_{g_t(x).\Gamma_x} + if_{h_t(x).\Gamma_x}\}\Big|_{t=0} = 0 .$$

In similar fashion we establish

$$L_{\xi_\phi} s_k = -y_\phi s_k . \tag{13.36}$$

(13.33) is now a routine consequence of (13.25) (13.35) (13.36).

□

We can give a rather explicit form to quantization if we suppose the polarization of (X,ω) to be linear (or, translation-invariant).

If Γ is a polarization of (V,Ω) then

$$x \in X \Rightarrow F_x = b_x^{\mathbb{C}}\Gamma \tag{13.37}$$

defines a translation-invariant polarization F of (X,ω) having the type of Γ (and all translation-invariant polarizations arise in this way). We call F the linear polarization of (X,ω) determined by Γ .

Let F be the positive linear polarization of (X,ω) determined by $\Gamma \in \mathrm{Lag}_+(V,\Omega)$. It is clear by differentiation of (13.11) that if $\phi \in C^1_F(X)$ then

$$x \in X \Rightarrow z_\phi(x) \in sp(V,\Omega)^{\mathbb{C}}_\Gamma \quad . \tag{13.38}$$

This is the key to simple proofs of our various results for this linear F . Note from Propositions 7.11 and 13.4 that

$$D_\phi s_f = \{\frac{1}{i\hbar}(\theta\xi_\phi) + \tfrac{1}{2}\mathrm{Tr}_\Gamma(z_\phi)\}s_f \tag{13.39}$$

for $f \equiv f_\Gamma$ and from Proposition 1.22 and 13.10 that

$$L_{\xi_\phi} s_k = -\mathrm{Tr}_\Gamma(z_\phi)s_k \tag{13.40}$$

for $k \equiv k_\Gamma$; whence a short proof of Proposition 13.12. We also have an explicit form for quantization as promised, thus:

<u>Proposition 13.13</u> : Quantization relative to the linear polarization F is given by

$$\delta^F_\phi(\psi.s_{f\otimes k}) = ((\frac{1}{i\hbar}(\phi+\theta\xi_\phi)-\tfrac{1}{2}\mathrm{Tr}_\Gamma(z_\phi))\psi + \{\phi,\psi\})s_{f\otimes k} \tag{13.41}$$

for $\phi \in C^1_F(X)$ and $\psi \in C(X)$.

<u>Proof</u>: Clear from (13.39) and (13.40).

□

<u>Remark 13.14</u> : We can describe $\Gamma_0(E'(P)^F \otimes K^F)$ as follows. It is readily verified that

$$X \in C_F(X) \Rightarrow Tr_\Gamma z_{\mathbf{x}} = 0 \quad . \tag{13.42}$$

Consequently the section $\psi \cdot s_{f\otimes k}$ is polarized iff

$$\xi_X \psi + \frac{1}{i\hbar}(\theta \xi_X)\psi = 0 \tag{13.43}$$

whenever $U \subset X$ is open and $X \in C_F(U)$. Choose a primitive $\theta^F \in \Gamma(X;F^0)$ for $\omega^{\mathbb{C}}$; this is possible since the linear F is automatically strongly regular. Cohomological triviality of X ensures the existence of $\lambda \in C(X)$ such that $\theta^F = \theta + d\lambda$. The condition for $\psi . s_{f\otimes k}$ to lie in $\Gamma_0(E'(P)^F \otimes K^F)$ can be rewritten in the form:

$$\psi \cdot s_{f\otimes k} \in \Gamma_0(E'(P)^F \otimes K^F) \Leftrightarrow e^{-\frac{1}{i\hbar}\lambda}\psi \in C_F(X) \ . \tag{13.44}$$

//

We ~~choose~~ close this section by introducing symplectic coordinates and giving concrete "p,q" expression to our explicit quantization formulae.

Let $(e_1,\ldots\ldots,e_m,f_1,\ldots\ldots,f_m)$ be a symplectic basis for (V,Ω); thus, if $1 \le i,j \le m$ then

$$\begin{aligned} &\Omega(e_i,e_j) = \Omega(f_i,f_j) = 0 \\ &\Omega(e_i,f_j) = \delta_{ij} \end{aligned} \tag{13.45}$$

Denoting by $(p_1,\ldots\ldots,p_m,q_1,\ldots\ldots,q_m)$ the dual basis for V^*, we have

$$b_x(e_i) = \frac{\partial}{\partial p_i}\Big|_x \quad , \quad b_x(f_i) = \frac{\partial}{\partial q_i}\Big|_x \tag{13.46}$$

for each $x \in X$ and

$$\xi_\phi = \sum_{j=1}^{m} \{ \frac{\partial\phi}{\partial q_j}\frac{\partial}{\partial p_j} - \frac{\partial\phi}{\partial p_j}\frac{\partial}{\partial q_j} \} \tag{13.47}$$

for each $\phi \in C(X)$. We can express z (13.9) relative to the chosen symplectic basis; it turns out that if $\phi \in C(X)$ then $z_\phi : X \to sp(V,\Omega)^{\mathbb{C}}$ corresponds to the function matrix

$$\begin{vmatrix} R_\phi & Q_\phi \\ P_\phi & S_\phi \end{vmatrix} \tag{13.48}$$

where : $(P_\phi)_{ij} = -\frac{\partial^2\phi}{\partial p_j \partial p_i}$, $(Q_\phi)_{ij} = \frac{\partial^2\phi}{\partial q_j \partial q_i}$, $(R_\phi)_{ij} = \frac{\partial^2\phi}{\partial p_j \partial q_i}$, $(S_\phi)_{ij} = -\frac{\partial^2\phi}{\partial q_j \partial p_i}$.

Let the prequantum form $\gamma = \frac{1}{2}\eta_*\alpha_0 + \frac{1}{i\hbar}\pi^*\theta$ on P be determined by the primitive

$$\theta = \sum_{j=1}^{m} p_j dq_j \tag{13.49}$$

of ω . We quote without proof the following examples of quantization relative to $(P,\gamma;F)$ where F is the linear polarization of (X,ω) determined by $\Gamma \in Lag_+(V,\Omega)$ as indicated.

<u>Remark 13.15</u> : Let Γ be the (real) polarization of (V,Ω) having basis $(e_1,\ldots\ldots,e_m)$. If $\phi \in C(X)$ then $\phi \in C_F(X)$ iff

$$1 \le j \le m \Rightarrow \frac{\partial\phi}{\partial p_j} = 0 \tag{13.50}$$

and $\phi \in C_F^1(X)$ iff

$$1 \le r,s \le m \Rightarrow \frac{\partial^2\phi}{\partial p_r \partial p_s} = 0 \ . \tag{13.51}$$

If $\phi \in C_F^1(X)$ then

$$Tr_\Gamma z_\phi = \sum_{j=1}^{m} \frac{\partial^2\phi}{\partial p_j \partial q_j} \ . \tag{13.52}$$

The section $\psi \cdot s_{f\otimes k}$ of $E'(P)^F \otimes K^F$ is polarized iff $\psi \in C_F(X)$. Quantization (13.41) takes the concrete form

$$\delta_{\phi}^{F}(\psi.s_f \otimes s_k) = \{(\frac{1}{i\hbar}(\phi - \Sigma p_j \frac{\partial\phi}{\partial p_j}) - \tfrac{1}{2}\Sigma \frac{\partial^2\phi}{\partial p_j \partial q_j})\psi - \Sigma \frac{\partial\phi}{\partial p_j} \frac{\partial\psi}{\partial q_j}\} \cdot s_f \otimes s_k \qquad (13.53)$$

for $\phi \in C_F^1(X)$ and $\psi \in C_F(X)$.

//

<u>Remark 13.16</u> : Let Γ be the (strictly positive) polarization of (V,Ω) having basis $(e_1 + if_1,\ldots\ldots,e_m + if_m)$. If $\phi \in C(X)$ then $\phi \in C_F(X)$ iff

$$1 \le j \le m \Rightarrow \frac{\partial\phi}{\partial p_j} + i\,\frac{\partial\phi}{\partial q_j} = 0 \qquad (13.54)$$

(Cauchy-Riemann) and $\phi \in C_F^1(X)$ iff

$$1 \le r,s \le m \Rightarrow \begin{cases} \dfrac{\partial^2\phi}{\partial p_r \partial p_s} = \dfrac{\partial^2\phi}{\partial q_r \partial q_s} \\[2ex] \dfrac{\partial^2\phi}{\partial p_r \partial q_s} + \dfrac{\partial^2\phi}{\partial q_r \partial p_s} = 0 \end{cases} \qquad (13.55)$$

If $\phi \in C_F^1(X)$ then

$$Tr_\Gamma z_\phi = i \sum_{j=1}^{m} \frac{\partial^2\phi}{\partial q_j \partial q_j} \quad . \qquad (13.56)$$

The section $\psi \cdot s_{f\otimes k}$ of $E'(P)^F \otimes K^F$ is polarized iff $\exp(+\frac{1}{2\hbar}\Sigma p_j^2)\cdot\psi \in C_F(X)$. Quantization (13.41) takes the concrete form

$$\delta_\phi^F(\psi\cdot s_f \otimes s_k) = \tag{13.57}$$

$$\{(\frac{1}{i\hbar}(\phi-\Sigma p_j \frac{\partial\phi}{\partial p_j}) - \frac{1}{2}i\Sigma \frac{\partial^2\phi}{\partial q_j^2})\psi + \Sigma(\frac{\partial\phi}{\partial q_j}\frac{\partial\psi}{\partial p_j} - \frac{\partial\phi}{\partial p_j}\frac{\partial\psi}{\partial q_j})\}\cdot s_f \otimes s_k$$

for $\phi \in C_F^1(X)$ and $\exp\{+\frac{1}{2\hbar}\Sigma p_j^2\}\cdot\psi \in C_F(X)$.

//

Remark 13.17 : We leave the account with the following observations. It turns out that the half-form pairing in Remark 13.15 gives rise to the Hilbert space $L^2(\mathbb{R}^m)$, that the half-form pairing in Remark 13.16 gives rise to the Hilbert space $F_0(\mathbb{C}^m)$, and that the half-form pairing between these two cases gives the transform of Bargmann [Bn1]. See the corresponding treatment in Rawnsley [Ry4].

§14. Complex Projective Spaces.

Of perhaps greater practical interest than the linear symplectic manifolds are the complex projective spaces, arising when one considers the energy surfaces of harmonic oscillators. We shall see that our scheme for geometric quantization provides a uniform treatment of harmonic oscillators irrespective of dimensional parity (as does the scheme due to Czyz [Cz]). This is in contrast with the Kostant scheme, which is unable to deal with the odd-dimensional harmonic oscillators (since the even dimensional complex projective spaces are neither integral nor metaplectic).

We begin with a brief review of some differential geometry of complex projective spaces. For further information consult Kobayashi & Nomizu [KN] and Wells [Ws]. V will in this section denote a complex vector space of complex dimension $m = n+1$ with $n > 0$.

We refer to the space of all complex lines in V as the projective space $\mathbb{P}(V)$ of V . $\mathbb{P}(V)$ is an n-dimensional complex manifold. Let $(z^1,\ldots\ldots,z^m)$ be a basis for V^* . For $1 \le j \le m$ define

$$U_j = \{\ell \in \mathbb{P}(V) \mid \ell \cap \ker z^j = 0\} \tag{14.1}$$

and for $1 \le k \le m$ define

$$w_j^k : U_j \to \mathbb{C} : \langle v \rangle \mapsto \frac{z^k(v)}{z^j(v)} \qquad (14.2)$$

where $\langle v \rangle$ denotes the complex line through $v \in V\backslash\{0\}$. Then $(w_j^1,\ldots\ldots,\hat{w}_j^j,\ldots\ldots,w_j^m)$ are standard holomorphic coordinates on U_j defined by $(z^1,\ldots\ldots,z^m)$ (and of course $w_j^j \equiv 1$).

Recall that on every complex manifold we have a natural bigrading

$$\Omega^r = \bigoplus_{p+q=r} \Omega^{p,q} \qquad (14.3)$$

of complex forms according to type, and a natural decomposition of exterior derivative

$$d = \partial + \bar{\partial} \qquad (14.4)$$

with $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$, $\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}$.

Equip V with a Hermitian inner product $\langle\cdot,\cdot\rangle$ having norm $|\cdot|$. Define

$$\hat{\phi} = \partial\bar{\partial} \log |\cdot|^2 \in \Omega^{1,1}(V) . \qquad (14.5)$$

Viewing the natural map $\nu_V : V\backslash\{0\} \to \mathbb{P}(V) : v \mapsto \langle v \rangle$ as a principal $\mathbb{C}^*$ bundle, $\hat{\phi}|V\backslash\{0\}$ is a basic form and so descends to define a type (1,1) form ϕ on $\mathbb{P}(V)$:

$$\nu_V^* \Phi = \hat{\Phi} . \tag{14.6}$$

If $(z^1,\ldots\ldots,z^m)$ is a $\langle\cdot,\cdot\rangle$-unitary basis for V^*, so that

$$|\cdot|^2 = |z_1|^2 + \ldots\ldots + |z_m|^2 \quad ,$$

then in terms of the standard holomorphic coordinates defined by $(z^1,\ldots\ldots,z^m)$ we have

$$\Phi|U_j = L_j^{-4}\{L_j^2 \sum_{k\neq j} dw_j^k \wedge \overline{dw_j^k} - \sum_{r,s\neq j} \bar{w}_j^r w_j^s \, dw_j^r \wedge \overline{dw_j^s}\} \tag{14.7}$$

where $L_j : U_j \to \mathbb{R}_+$ is defined by

$$L_j^2 = 1 + \sum_{k\neq j} |w_j^k|^2 \quad . \tag{14.8}$$

We now turn to a discussion of (holomorphic) complex line bundles on $\mathbb{P}(V)$. The group of (isomorphismclasses of) holomorphic line bundles on $\mathbb{P}(V)$ is infinite cyclic, generated by the hyperplane section bundle; the same is true of the group of (isomorphism classes of) complex line bundles on $\mathbb{P}(V)$. These line bundles on $\mathbb{P}(V)$ are determined (up to isomorphism) by their real Chern classes.

Let us define the hyperplane section bundle $\pi_H : H_V \to \mathbb{P}(V)$. The total space H_V is the set of all complex linear functionals on complex lines in V and the projection π_H assigns to each linear functional the line on which it acts - thus, $\pi_H^{-1}(\ell) = \ell^*$ for $\ell \in \mathbb{P}(V)$. The hyperplane section bundle is naturally a holomorphic line bundle; we can conveniently describe its holomorphic structure as follows. Each $f \in V^*$ yields (by restriction) an element $f|\ell$ of $\ell^* = \pi_H^{-1}(\ell)$ for every $\ell \in \mathbb{P}(V)$; thus we may define $s_f \in \Gamma(\mathbb{P}(V);H_V)$ by

$$s_f : \mathbb{P}(V) \to H_V : \ell \mapsto f|\ell \quad . \tag{14.9}$$

Then there is a unique holomorphic structure in π_H such that s_f is holomorphic for all $f \in V^*$; moreover, $\{s_f | f \in V^*\}$ exhausts the space of (global) holomorphic sections of π_H . If U is an open subset of $\mathbb{P}(V)$ then $\mathcal{O}(U;H_V)$ denotes the space of holomorphic sections of $\pi_H : H_V \to \mathbb{P}(V)$ over U . Thus we have a natural isomorphism

$$s : V^* \to \mathcal{O}(\mathbb{P}(V);H_V) : f \mapsto s_f \quad . \tag{14.10}$$

More generally, if $P^k(V)$ denotes the space of homogeneous polynomials of degree k on V , then we have a natural isomorphism

$$P^k(V) \to \mathcal{O}(\mathbb{P}(V);(H_V)^k) \tag{14.11}$$

where $(H_V)^k = H_V \otimes \ldots\ldots \otimes H_V$ (k factors), for each $k \in \mathbb{N} \cup \{0\}$. If $k < 0$ then

$$\mathcal{O}(\mathbb{P}(V) \; ; \; (H_V)^k) = 0 \quad . \tag{14.12}$$

Consider next Hermitian structures in π_H. Equip V with the Hermitian inner product $\langle\cdot,\cdot\rangle$ having norm $|\cdot|$. If $\ell \in \mathbb{P}(V)$, then $\langle\cdot,\cdot\rangle$ induces a Hermitian inner product $\langle\cdot,\cdot\rangle_\ell$ on $\ell^* = \pi_H^{-1}(\ell)$ by restriction and dualization. In this way, $\langle\cdot,\cdot\rangle$ gives rise to a Hermitian structure $\langle\cdot,\cdot\rangle_H$ in the hyperplane section bundle $\pi_H : H_V \to \mathbb{P}(V)$. Let $(z^1,\ldots\ldots,z^m)$ be a $\langle\cdot,\cdot\rangle$-unitary basis for V^* and for $1 \le j \le m$ define $s_j = s_{zj} \in \mathcal{O}(\mathbb{P}(V);H_V)$ via (14.10); then over $U_j \subset \mathbb{P}(V)$ we have

$$\langle s_j,s_j\rangle_H = L_j^{-2} \tag{14.13}$$

with L_j given by (14.8).

Recall (from Wells [Ws]) the following. Any Hermitian holomorphic line bundle (more generally, vector bundle) admits a unique connexion which is compatible with both the Hermitian structure and the holomorphic structure. If s is a zero-free holomorphic section over the set U and

$$\alpha_s = \partial(\log\langle s,s\rangle) \tag{14.14}$$

then this connexion ∇ is given by

$$\xi \in \mathfrak{X}(U) \Rightarrow \nabla_\xi s = \alpha_s(\xi)s \tag{14.15}$$

and has curvature ρ given by

$$\rho = \bar\partial\alpha_s \ . \tag{14.16}$$

Our next task is to give an explicit formula for the unique connexion ∇ in π_H which is compatible both with the natural holomorphic structure and with the Hermitian structure $\langle\cdot,\cdot\rangle_H$. If $(z^1,\ldots\ldots,z^m)$ is a $\langle\cdot,\cdot\rangle$ -unitary basis for V^* and $s_j = s_{z^j} \in \mathcal{O}(\mathbb{P}(V);H_V)$ then from (14.13) (14.14) (14.15) we deduce

$$\xi \in \mathfrak{X}(U_j) \Rightarrow \nabla_\xi s_j = -L_j^{-2} \sum_{k \neq j} \bar{w}_j^k \, dw_j^k(\xi) s_j \tag{14.17}$$

and (14.7) (14.16) in addition imply that ∇ has curvature precisely Φ :

$$[\nabla_\xi,\nabla_\eta]s - \nabla_{[\xi,\eta]}s = \Phi(\xi,\eta)s \tag{14.18}$$

for $\xi,\eta \in \mathfrak{X}(\mathbb{P}(V))$ and $s \in \Gamma(\mathbb{P}(V);H_V)$.

Observe that as a consequence of (14.18) the hyperplane section bundle has real Chern class

$$c[H_V]^{\mathbb{R}} = [- \frac{\Phi}{2\pi i}] . \tag{14.19}$$

More generally, if $k \in \mathbb{Z}$ then

$$c[(H_V)^k]^{\mathbb{R}} = [- k \frac{\Phi}{2\pi i}] . \tag{14.20}$$

Let F denote the bundle of anti-holomorphic tangents to $\mathbb{P}(V)$. The canonical bundle K^F is then the bundle of holomorphic n-forms on $\mathbb{P}(V)$. From our earlier comments we know that K^F is isomorphic to some tensor power of H_V ; precisely, K^F is isomorphic to $(H_V)^{-m} = (H_V^*)^{n+1}$, as one checks by a careful comparison of transition functions. Consequently

$$c[K^F]^{\mathbb{R}} = [m \frac{\Phi}{2\pi i}] \quad . \tag{14.21}$$

This concludes our review of the familiar differential geometry of the complex projective spaces; we turn next to their symplectic geometry.

For $E \in \mathbb{R}_+$ we define

$$\omega_E = iE\Phi \tag{14.22}$$

$(\mathbb{P}(V),\omega_E)$ is a symplectic manifold of which the antiholomorphic tangent bundle F is a positive polarization. Up to scalar multiples, ω_E is the fundamental Kähler form of the Fubini-Study metric on $\mathbb{P}(V)$. We may therefore refer to ω_E as the Fubini-Study symplectic form of energy E .

Fix a $\langle\cdot,\cdot\rangle$-unitary basis $(z^1,.....,z^m)$ of V^* . Relabelling for convenience, we have the standard holomorphic coordinates $(w^1,.....,w^n)$ on $U = U_j \subset \mathbb{P}(V)$ and the length function $L = L_j : U_j \to \mathbb{R}_+$. We shall employ this notation henceforth.

We find it convenient to record the local symplectic structure of $(\mathbb{P}(V),\omega_E)$.

<u>Proposition 14.1</u> : For $1 \leq k \leq n$ define real-valued functions p^k , q^k on U by

$$\sqrt{2E}\,\frac{w^k}{L} = p^k + iq^k \quad . \tag{14.23}$$

Then

$$\omega_E|U = \sum_{k=1}^{n} dp^k \wedge dq^k \quad . \tag{14.24}$$

Thus $(p^1,\ldots\ldots,p^n,q^1,\ldots\ldots,q^n)$ are symplectic coordinates for $(\mathbb{P}(V),\omega_E)$ over $U \subset \mathbb{P}(V)$.

<u>Proof</u>: A straightforward deduction from the local expression (14.7) for Φ . □

When we come (shortly) to quantize $(\mathbb{P}(V),\omega_E)$, we shall require an explicit formula for Lie differentiation in the canonical bundle K^F . This we present next.

In terms of our standard holomorphic coordinate system $(w^1,\ldots\ldots,w^n)$ on U , define

$$k = dw^1 \wedge\ldots\ldots\wedge dw^n \in \mathcal{O}(U;K^F) \quad . \tag{14.25}$$

<u>Proposition 14.2</u> : If $\phi \in C^1_F(\mathbb{P}(V))$ then

$$L_{\xi_\phi} k = \frac{L^2}{iE}\left(m\bar{w}^b \frac{\partial\phi}{\partial\bar{w}^b} + (\delta^{rs} + w^r\bar{w}^s)\frac{\partial^2\phi}{\partial w^r \partial\bar{w}^s}\right)k \tag{14.26}$$

where δ^{rs} is the Kronecker delta and the indices b,r,s are summed over $\{1,\ldots\ldots,n\}$.

<u>Proof</u>: Proposition 14.1 leads us to the following relationship between the holomorphic and symplectic structures on $(\mathbb{P}(V),\omega_E)$: if $\phi \in C(U)$ then

$$dw^r(\xi_\phi) = \frac{L^2}{iE} \sum_{s=1}^{n} (\delta^{rs} + w^r\bar{w}^s) \frac{\partial\phi}{\partial\bar{w}^s} \quad . \tag{14.27}$$

Now, if $\phi \in C_F^1(\mathbb{P}(V))$ then

$$L_{\xi_\phi} k = \sum_{r=1}^{n} dw^1 \wedge \ldots\ldots \wedge L_{\xi_\phi} dw^r \wedge \ldots\ldots \wedge dw^n$$

$$= \sum_{r=1}^{n} dw^1 \wedge \ldots\ldots \wedge d(dw^r(\xi_\phi)) \wedge \ldots\ldots \wedge dw^n$$

and the holomorphicity of $\{\phi,w^r\} = dw^r(\xi_\phi)$ implies that

$$d(dw^r(\xi_\phi)) = \sum_{s=1}^{n} \frac{\partial(dw^r(\xi_\phi))}{\partial w^s} dw^s$$

whence

$$L_{\xi_\phi} k = \{ \sum_{k=1}^{n} \frac{\partial}{\partial w^k} (dw^k(\xi_\phi))\} k \quad .$$

This expression is evaluated with the aid of (14.27) and yields (14.26).

□

We are now free to consider the geometric quantization of $(\mathbb{P}(V),\omega_E)$.

Since $H^2(\mathbb{P}(V);\mathbb{Z})$ is infinite cyclic (generated by the Chern class $c[H_V]$ of the hyperplane section bundle) the group $T(\mathbb{P}(V),\omega_E)$ of equivalence classes of Mp^c structures for $(T\mathbb{P}(V),\omega_E)$ is also infinite cyclic. As regards quantizability,

<u>Proposition 14.3</u> : We have the formula

$$[\frac{\omega_E}{h}] + \tfrac{1}{2}\, c_1(T\,\mathbb{P}(V),\omega_E)^{\mathbb{R}} = [-(\frac{E}{\hbar} - \frac{m}{2})\,\frac{\Phi}{2\pi i}] \quad . \tag{14.28}$$

Thus, $(\mathbb{P}(V),\omega_E)$ is quantizable iff

$$E = (N + \tfrac{1}{2}m)\hbar \quad , \quad N \in \mathbb{Z} \quad . \tag{14.29}$$

<u>Proof</u>: (14.28) comes from (14.21) and (14.22). The quantization condition (14.29) is immediate from (14.28) in view of (14.20) and our above remarks on $H^2(\mathbb{P}(V);\mathbb{Z})$.

□

Assume from now on that the quantization condition (14.29) holds.

Since $\mathbb{P}(V)$ is simply-connected it follows that $H^1(\mathbb{P}(V);U(1))$ is trivial. Consequently all prequantized Mp^c structures for $(\mathbb{P}(V),\omega_E)$ are equivalent. Choose and fix the prequantized Mp^c structure (P,γ) .

Refer to Remark 12.8. We have canonical structures of holomorphic line bundle on K^F , $P(n)$, $E'(P)^F$, $E'(P)^F \otimes K^F$, and a canonical isomorphism of holomorphic line bundles

$$(E'(P)^F \otimes K^F)^2 \to P(n) \otimes K^F \quad . \tag{14.30}$$

From (14.20) (14.21) (14.22) it is a simple matter to deduce that we have holomorphic isomorphisms

$$\begin{aligned} K^F &= H_V^{-m} & P(n) &= H_V^{2N+m} \\ E'(P)^F &= H_V^{N+m} & E'(P)^F \otimes K^F &= H_V^N \end{aligned} \tag{14.31}$$

These isomorphisms are unique modulo $\overset{\circ}{\mathbb{C}}$ and may be chosen to make (14.30) correspond to the standard isomorphism

$$(H_V^N)^2 \to H_V^{2N+m} \otimes H_V^{-m} \, . \tag{14.32}$$

<u>Remark 14.4</u> : In view of (14.11) (14.12) the space of polarized (holomorphic) sections of $E'(P)^F \otimes K^F$ over $\mathbb{P}(V)$ can be identified with $P^N(V)$ if $N \geq 0$ and is zero if $N < 0$. Thus we have the practical quantization condition

$$E = (N + \tfrac{1}{2}m)\hbar \quad , \quad N \in \mathbb{N} \cup \{0\} \, . \tag{14.33}$$

//

Let us now develop an explicit formula for quantization δ^F of $(\mathbb{P}(V),\omega_E)$ relative to $(P,\gamma;F)$.

By means of (14.31) we pass across the various operators involved

$$(E'(P)^F \otimes K^F)^2 \to P(n) \otimes K^F \quad . \tag{14.30}$$

From (14.20) (14.21) (14.22) it is a simple matter to deduce that we have holomorphic isomorphisms

$$\begin{aligned} K^F &= H_V^{-m} & P(n) &= H_V^{2N+m} \\ E'(P)^F &= H_V^{N+m} & E'(P)^F \otimes K^F &= H_V^N \end{aligned} \tag{14.31}$$

These isomorphisms are unique modulo $\mathring{\mathbb{C}}$ and may be chosen to make (14.30) correspond to the standard isomorphism

$$(H_V^N)^2 \to H_V^{2N+m} \otimes H_V^{-m} \ . \tag{14.32}$$

<u>Remark 14.4</u> : In view of (14.11) (14.12) the space of polarized (holomorphic) sections of $E'(P)^F \otimes K^F$ over $\mathbb{P}(V)$ can be identified with $P^N(V)$ if $N \geq 0$ and is zero if $N < 0$. Thus we have the practical quantization condition

$$E = (N + \tfrac{1}{2}m)\hbar \quad , \quad N \in \mathbb{N} \cup \{0\} \ . \tag{14.33}$$

//

Let us now develop an explicit formula for quantization δ^F of $(\mathbb{P}(V),\omega_E)$ relative to $(P,\gamma;F)$.

By means of (14.31) we pass across the various operators involved in quantization to the appropriate tensor powers of the hyperplane section bundle. We shall denote by $s^k \in \mathcal{O}(U;H_V^K)$ the kth. tensor power of the holomorphic section $s = s_j$ of H_V over $U = U_j$.

$(P(n),\nabla^\gamma)$ becomes the (2N+m)th. power H_V^{2N+m} equipped with the unique connexion ∇^{2N+m} which is compatible both with the canonical holomorphic structure and with the Hermitian structure coming from $\langle\cdot,\cdot\rangle_H$ in H_V .

<u>Proposition 14.5</u> : $\nabla^\gamma = \nabla^{2N+m}$ is given on $s^{2N+m} \in \mathcal{O}(U;P(n))$ as follows. If $\xi \in \mathfrak{X}(U)$ then

$$\nabla_\xi^\gamma s^{2N+m} = -(2N+m)L^{-2}\sum_{r=1}^{n} \bar{w}^r dw^r(\xi)s^{2N+m} \quad . \tag{14.34}$$

If $\phi \in C_F^1(U)$ then

$$\nabla_{\xi_\phi}^\gamma s^{2N+m} = -(2N+m)\frac{L^2}{iE}\sum_{r=1}^{n} \bar{w}^r \frac{\partial\phi}{\partial\bar{w}^r} s^{2N+m} \quad . \tag{14.35}$$

<u>Proof</u>: (14.34) is the (2N+m)th. tensor power of (14.17). (14.35) follows from (14.35) by (14.27).

□

Observe that we are now in the position covered by Remark 12.6 : we have identified both $(P(n),\nabla^\gamma) = (H_V^{2N+m},\nabla^{2N+m})$ and the square-root $E'(P)^F \otimes K^F = H_V^N$ of $P(n) \otimes K^F = H_V^{2N+m} \otimes H_V^{-m} = H_V^{2N}$. Thus:

Proposition 14.6 : If $\phi \in C^1_F(U)$ then quantization

$$\delta^F_\phi = D^F_\phi + \frac{1}{i\hbar}\phi \tag{14.36}$$

is determined by

$$D^F_\phi s^N = \frac{L^2}{(2N+m)i\hbar}\{(\delta^{rs}+w^r\bar{w}^s)\frac{\partial^2\phi}{\partial w^r \partial\bar{w}^s} - 2Nw^b\frac{\partial\phi}{\partial\bar{w}^b}\}s^N \tag{14.37}$$

where the indices b,r,s are summed over $\{1,\ldots\ldots,n\}$.

Proof: (14.37) comes from (14.26) and (14.35) via the Leibnitz rule (12.14) applied to (14.30) (14.32).

□

Remark 14.7 : (14.37) only describes the effect of quantization on the special holomorphic section $s^N = s^N_j$ of H^N_V over U_j . Referring to (14.11), the general form of a holomorphic section of H^N_V restricted to U_j is

$$\lambda(w^1_j)^{r_1} \ldots\ldots (w^j_j)^{r_j} \ldots\ldots (w^m_j)^{r_m} s^N_j \tag{14.38}$$

where $\lambda \in \mathbb{C}$ and $r_1 + \ldots\ldots + r_m = N$ (coming from the degree N polynomial $\lambda z_1^{r_1} \ldots\ldots z_m^{r_m}$ on V) . It is clear how to quantize (14.38) using (14.37).

//

§15. Prospects.

Many topics remain to be detailed, both in the abstract theories of the metaplectic representation and Mp^c structures and in their implications for geometric quantization. We mention just two of these topics before passing on to a consideration of other matters.

In discussing Mp^c structures, we defined half-forms for positive polarizations and constructed their pairing in the regular case. As mentioned in our discussion, Blattner & Rawnsley [BR] have shown how to define half-forms for arbitrary polarizations. It is natural to ask how we should pair the half-forms which come from an arbitrary pair of polarizations; as yet there is no firm answer to this question.

The examples of symplectic manifolds which we have chosen to illustrate our methods have all been simply-connected, admitting a unique prequantized Mp^c structure (up to equivalence). It is natural to ask how geometric quantization looks when inequivalent prequantized Mp^c structures are available. The simplest example to consider is the punctured plane with its flat symplectic structure; this will have a full circle set of inequivalent prequantized Mp^c structures, as $\pi_1(\mathbb{R}^2\setminus\{0\})$ is infinite cyclic. More interesting is the study of compact quotients of bounded domains by fixed-point-free properly-discontinuous groups of biholomorphisms. In addition to their admitting inequivalent prequantized Mp^c structures, these compact symplectic manifolds are (in favourable cases) quantizable but not metaplectic.

It is hoped that a discussion of these (and other) topics will appear in due course.

We have seen that every symplectic manifold admits symplectic spinors defined by Mp^c structures and that we are consequently able to construct half-forms and their pairing for (positive) polarizations of an arbitrary symplectic manifold. The question which now confronts us is whether or not these symplectic spinors and half-forms are of use in a given symplectic context.

Our account of geometric quantization provides one context in which an important rôle is played by the symplectic spinors arising from an Mp^c structure. We now turn to a brief consideration of another such context: the symbolic theory of operators.

The space available to us at present does not allow an adequate discussion of the extensive background to this topic and the contributions made by such authors as Boutet de Monvel, Colin de Verdière, Duistermaat, Guillemin, Hörmander, Melin, Menikoff, Sjöstrand, Sternberg, Weinstein. As principal reference for the following discussion we cite Boutet de Monvel & Guillemin (concisely, BdM & G) [BG] .

We shall be content here to achieve the following two-fold aim (briefly and without details): first, to describe the setting for [BG] Theorem 7.5 (which BdM & G themselves consider to be perhaps their main result in [BG]) and second, to describe how the use of Mp^c structures leads to a proof of this result (in modified form). Again, it is hoped that this will appear more fully at a later stage.

Let M be a manifold. The cotangent bundle T^*M is naturally a symplectic manifold. If $0_M \subset T^*M$ denotes the zero section then $\mathbb{R}_+$ acts naturally (fibrewise) on $X = T^*M\backslash 0_M$. Let Σ be a homogeneous isotropic submanifold of X . BdM & G define spaces $I^a(M,\Sigma)$ of generalized half-forms and assert the existence of symbol maps

$$\sigma : I^a(M,\Sigma) \to S^a(\Sigma) \tag{15.1}$$

where $S^a(\Sigma)$ is the space of a-homogeneous sections of a certain vector bundle $\mathrm{Spin}(\Sigma)$ over Σ . $\mathrm{Spin}(\Sigma)$ reduces to a bundle of (metalinear) half-forms for Σ when Σ is Lagrangian, but is otherwise infinite-dimensional.

Suppose N to be another manifold. Write $Y = T^*N\backslash 0_N$ and $Z = T^*(M\times N)\backslash 0_{M\times N}$. Let Γ be a homogeneous Lagrangian submanifold of Z . Assume that the clean intersection hypotheses of [BG] §7 are satisfied. We then have an (immersed) isotropic submanifold $\Gamma\circ\Sigma$ of Y .

BdM & G assert that a c-homogeneous half-form γ on Γ determines a map

$$\gamma\circ(\cdot) : S^a(\Sigma) \to S^{a+c-d}(\Gamma\circ\Sigma) \tag{15.2}$$

where d is determined purely from the clean intersection data.

[BG] Theorem 7.5 asserts that a generalized half-form $k \in I^c(M\times N,\Gamma)$ gives rise to a map

$$K : I^a(M,\Sigma) \to I^{a+c-d}(N,\Gamma\circ\Sigma) \tag{15.3}$$

which satisfies the composition property

$$u \in I^a(M,\Sigma) \Rightarrow \sigma(Ku) = \sigma(k)\circ\sigma(u) \tag{15.4}$$

relative to (15.2) with $\gamma = \sigma(k)$.

Of course, this procedure cannot always be carried through. BdM & G assume M and N to be equipped with metalinear structures; thus, T^*M and T^*N are provided with metaplectic structures. Unfortunately, this assumption does not suffice when the isotropic submanifold Σ is not Lagrangian. To see why this should be so, we must take a closer look at the BdM & G procedure.

BdM & G denote by $(B\times S)\Sigma$ the principal $G\ell \times Sp$ bundle over Σ whose fibre over $x \in \Sigma$ is the set of pairs (b,s) with b a frame of $T_x\Sigma$ and s a frame of $T_x\Sigma^{\perp}/T_x\Sigma$. They assert that when M is metalinear (so that X is metaplectic) $(B\times S)\Sigma$ lifts to a four-fold cover by a principal $M\ell \times Mp$ bundle P , to which $\mathrm{Spin}\ (\Sigma)$ is associated via the half-form representation of $M\ell$ tensored with the metaplectic representation of Mp . Factoring P by $M\ell$ would give $T\Sigma^{\perp}/T\Sigma$ a metaplectic structure, in contradiction to Remark 8.13 unless Σ is metalinear. Even when Σ is metalinear it is clear from Proposition 6.10 that an $M\ell \times Mp$ lift P of $(B{\times}S)\Sigma$ is not associated to a metaplectic structure for X. We thus encounter problems in the construction of Spin (Σ).

When we attempt to construct the various symbol maps we meet further problems; these arise from the BdM & G description (in [BG] §4) of our space $(E')^L$ for L an isotropic subspace of (V,Ω) with $L \neq 0 \neq L^{\perp}/L$. BdM & G identify $(E')^L$ with E_L' tensored by a (metalinear) square-root of $\Lambda^r L^{\mathbb{C}}$; this identification they establish by means of a metaplectic lift of ρ_L (1.73). However, as we saw in Proposition 6.10, ρ_L does not lift to the level of Mp.

These difficulties can be circumvented by the use of Mp^c structures in place of Mp structures. Moreover, the resulting scheme will apply to any manfiolds M and N without the assumption of metalinearity.

The correct linear formalism is crucial to our scheme, and the key result here is a version of [BG] Proposition 6.5. Briefly, we replace half-forms (which depend on a choice of metalinear structure globally) by half-densities (which are naturally present) and (when Σ is not Lagrangian) make use of the morphism

$$S_L : E \to E_L \otimes \mathcal{D}^{-\frac{1}{2}}(L) \qquad (15.5)$$

of representations of $Mp^c(V,\Omega;L)$ which comes by the duality in Proposition 3.6 from (6.13) which may be viewed as an isomorphism

$$R_L : (E')^L \to E'_L \otimes \mathcal{D}^{\frac{1}{2}}(L) \tag{15.6}$$

of representations of $Mp^c(V,\Omega;L)$.

Now let M be an arbitrary (smooth) manifold and let $X = T^*M\backslash 0_M$. Remark 8.8 tells us that TX has a canonical (neutral) class of Mp^c structures; choose and fix a member P of this class. Let Σ be a homogeneous isotropic submanifold of X . We may take as our version of $I^a(M,\Sigma)$ the corresponding space of generalized half-densities. Suppose Σ is not Lagrangian; by restriction to Σ and passage to the normal (as in Proposition 8.10) P gives to $T\Sigma^{\perp}/T\Sigma$ an Mp^c structure P_Σ , and we may take as our version of $\mathrm{Spin}(\Sigma)$ the vector bundle over Σ associated to P_Σ with fibre $E_L \otimes \mathcal{D}^{\frac{1}{2}}(L)$ (modelling $T\Sigma$ on L) :

$$\mathrm{Spin}(\Sigma) = E_L(P_\Sigma) \otimes \mathcal{D}^{\frac{1}{2}}(\Sigma) \quad . \tag{15.7}$$

If Σ is Lagrangian then $\mathrm{Spin}(\Sigma) = \mathcal{D}^{\frac{1}{2}}(\Sigma)$.

Now suppose N to be another manifold and let $Y = T^*N\backslash 0_N$, $Z = T^*(M\times N)\backslash 0_{M\times N}$. Fix a neutral Mp^c structure Q for Y . Let Γ be a homogeneous Lagrangian submanifold of Z , and assume the clean intersection hypotheses of [BG]§7.

Our version of [BG] Proposition 6.5 (applied as BdM & G apply theirs in [BG] §7) will enable us to construct symbol maps $\sigma : I^a(M,\Sigma) \to S^a(\Sigma)$, $\sigma : I^b(N,\Gamma o\Sigma) \to S^b(\Gamma o\Sigma)$, $\sigma : I^c(M\times N,\Gamma) \to S^c(\Gamma)$ and show that a half-density $\gamma \in S^c(\Gamma)$ determines a map

$$\gamma o(\cdot) : S^a(\Sigma) \to S^{a+c-d}(\Gamma o\Sigma) \tag{15.8}$$

corresponding to (15.2).

The appropriate version of [BG] Theorem 7.5 will then state that a generalized half-density $k \in I^c(M\times N,\Gamma)$ gives rise to a map

$$K : I^a(M,\Sigma) \to I^{a+c-d}(N,\Gamma o\Sigma) \tag{15.9}$$

satisfying the composition property

$$u \in I^a(M,\Sigma) \Rightarrow \sigma(Ku) = \sigma(k)o\sigma(u) \tag{15.10}$$

in general modulo Maslov factors. We shall not discuss Maslov factors; see Hörmander [Hr] for these.

We should point out that our proposed scheme for the construction of symbols is as yet in its germinal stages. Since it deals with half-densities in place of half-forms, it represents a return to the philosophy of Hörmander. The consequent reappearance of Maslov factors could be viewed as a disadvantage; indeed, one of the reasons for working with half-forms is that they absorb Maslov factors. However, it would appear that our proposal has several advantages over the BdM & G procedure. Thus:

(a) It circumvents the aforementioned technical difficulties in the construction of the various symbol maps.

(b) It removes the metalinear restriction on M and N.

(c) In the BdM & G procedure one must choose metalinear structures for M and N. Up to equivalence, these are parametrized by $H^1(M;\mathbb{Z}_2)$ and $H^1(N;\mathbb{Z}_2)$ and in general none is preferred. In our proposed scheme, however, we are presented with natural choices of Mp^c structure for X and Y (those which are neutral). Of course, we could use other Mp^c structures in constructing symbols; account must then be made of their corresponding complex line bundles.

References.

[Bn1] Bargmann, V. : "On a Hilbert Space of Analytic Functions and an Associated Integral Transform. Part I." Comm. Pure Appl. Math. 14 (1961) 187-214.

[Bn2] Bargmann, V. : "On a Hilbert Space of Analytic Functions and an Associated Integral Transform. Part II." Comm. Pure Appl. Math. 20 (1967) 1-101.

Blattner, R.J. : "Quantization and Representation Theory." A.M.S. Proc. Symp. Pure Math. 26 (1974) 147-165.

[Br] Blattner, R.J. : "The metalinear geometry of non-real polarizations." Springer-Verlag Lecture Notes in Mathematics 570 (1977) 11-45.

[BR] Blattner, R.J. & Rawnsley, J.H. : "A cohomological construction of half-forms for non-positive polarizations." Warwick preprint (1983).

[BG] Boutet' de Monvel, L. & Guillemin, V. : "The Spectral Theory of Toeplitz Operators." Princeton Annals of Mathematics Study 99 (1981).

[Cr] Cartier, P. : "Quantum Mechanical Commutation Relations and Theta Functions." A.M.S. Proc. Symp. Pure Math. 9 (1966) 361-383.

[Cz] Czyz, J. : "On some approach to geometric quantization." Springer-Verlag Lecture Notes in Mathematics 676 (1978) 315-328.

Duistermaat, J.J. & Guillemin, V.W. : "The Spectrum of Positive Elliptic Operators and Periodic Bicharacteristics" Inventiones Mathematicae 29 (1975) 39-79.

[FH] Forger, M. & Hess, H. : "Universal Metaplectic Structures and Geometric Quantization." Comm. Math. Phys. 64 (1979) 269-278.

[GP] Greub, W. & Petry, H.R. : "On the lifting of structure groups." Springer-Verlag Lecture Notes in Mathematics 676 (1978) 217-246.

[GS] Guillemin, V. & Sternberg, S. : "Geometric Asymptotics." A.M.S. Mathematical Surveys 14 (1977).

Guillemin, V. & Sternberg, S. : "The Metaplectic Representation, Weyl Operators and Spectral Theory." Jour. Func. Anal. 42 (1981) 128-225.

[Hs] Hess, H. : "On a geometric quantization scheme generalizing those of Kostant-Souriau and Czyz." Proc. Diff. Geom. Meth. Phys. Clausthal (1978).

[HK] Hess, H. & Krausser, D. : "Lifting classes of principal bundles." Berlin preprint (1977).

[Hr] Hörmander, L. : "Fourier Integral Operators. I." Acta Math. 127 (1971) 79-183.

[He] Howe, R. : "On the rôle of the Heisenberg group in harmonic analysis." Bull. of the A.M.S. 3 (1980) 821-843.

Hurt, N.E. : "Geometric Quantization in Action." Reidel; Mathematics and Its Applications 8 (1983).

[In] Itzykson, C. : "Remarks on Boson Commutation Rules." Comm. Math. Phys. 4 (1967) 92-122.

[Ki] Kobayashi, S. : "Irreducibility of certain unitary representations." Jour. Math. Soci. Japan 20 (1968) 638-642.

[KN] Kobayashi, S. & Nomizu, K. : "Foundations of Differential Geometry." John Wiley Interscience. Volume I (1963) & Volume II (1969).

[Kt1] Kostant, B. : " Quantization and Unitary Representations. Part I : Prequantization." Springer-Verlag Lecture Notes in Mathematics 170 (1970) 87-208.

[Kt2] Kostant, B. : "Symplectic Spinors." Academic Press, Symposia Mathematica 14 (1974).

[Kt3] Kostant, B. : "On the Definition of Quantization." Colloques Internationaux du CNRS 237 (1975).

[Pn] Plymen, R.J. : "The Weyl Bundle." Jour. Func. Anal. 49 (1982) 186-197.

[Ry1] Rawnsley, J.H. : "On the cohomology groups of a polarization and diagonal quantization." Tran. of the A.M.S. 230 (1977) 235-255.

[Ry2] Rawnsley, J.H. : "On the Pairing of Polarizations." Comm. Math. Phys. 58 (1978) 1-8.

[Ry3] Rawnsley, J.H. : "Flat partial connections and holomorphic structures in C^∞ vector bundles." Proc. of the A.M.S. 73 (1979) 391-397.

[Ry4] Rawnsley, J.H. : "A nonunitary pairing of polarizations for the Kepler problem." Tran. of the A.M.S. 250 (1979) 167-180.

[Ry5] Rawnsley, J.H. : "The Bargmann-Segal approach to symplectic spinors and half-forms for Mp^c structures." Warwick preprint (1983).

[Rℓ] Rieffel, M.A. : "On the Uniqueness of the Heisenberg Commutation Relations." Duke Math. Jour. 39 (1972) 745-752.

[Sℓ] Segal, I.E. : "Mathematical Problems of Relativistic Physics." A.M.S. Lectures in Applied Mathematics, Volume II (1963).

Segal, I.E. : "Transforms for operators and symplectic automorphisms over a locally compact abelian group." Math. Scan. 13 (1963) 31-43.

Shale, D. : "Linear Symmetries of Free Boson Fields." Tran. of the A.M.S. 103 (1962) 149-167.

[SW] Sternberg, S. & Wolf, J.A. : "Hermitian Lie algebras and Metaplectic representations. I." Tran. of the A.M.S. 238 (1978) 1-43.

Souriau, J.M. : "Structure des Systemes Dynamiques." Dunod, Paris (1970).

[Wℓ] Weil, A. : "Sur certains groupes d'opérateurs unitaires." Acta Math. 111 (1964) 143-211.

[Wn] Weinstein, A. : "Lectures on Symplectic Manifolds." A.M.S. CBMS 29 (1977).

[Ws] Wells, R.O. (Jr.) : "Differential Analysis on Complex Manifolds." Springer-Verlag Graduate Texts in Mathematics 65 (1980).

[We] Woodhouse, N. : "Geometric Quantization." Oxford University Press (1980).

II